**Solutions to sheet 2**

1. (a)
$$f_x = e^{x-y} + e^{y-x}, \qquad f_y = -e^{x-y} - e^{y-x}.$$

   (b)
$$g_x = z\cos(x-y), \quad g_y = -z\cos(x-y), \quad g_z = \sin(x-y).$$

   (c)
$$\rho_\theta = e^{\theta+\phi}(\cos(\theta-\phi) - \sin(\theta-\phi)), \qquad \rho_\phi = e^{\theta+\phi}(\cos(\theta-\phi) + \sin(\theta-\phi)).$$

   (d)
$$r_x = \frac{\sin(y)(\cos(x) + x\sin(x))}{y\cos^2(x)}, \qquad r_y = \frac{x(y\cos(y) - \sin(y))}{y^2\cos(x)}.$$

   (e)
$$h_x = 2xy + 3x^2 + y^2, \qquad h_y = x^2 + 3y^2 + 2yx.$$

   (f)
$$k(s,u) = e^{u\log s + s\log u}, \qquad k_s = (u/s + \log u)s^u u^s, \qquad k_u = (s/u + \log s)s^u u^s.$$

2.
$$f_x = 2xy, \quad f_y = x^2, \qquad g_x = \frac{2}{x}, \quad g_y = \frac{1}{y}.$$

3. Here we have to use the geometrical definition of partial derivative. For the function $z = f(x,y) = \sqrt{4-x^2-y^2}$ we have that
$$f_x = \frac{-x}{\sqrt{4-x^2-y^2}}, \qquad f_y = \frac{-y}{\sqrt{4-x^2-y^2}}.$$
At the point $(x,y,z) = (1,1,\sqrt{2})$ the value of the partial derivatives is
$$f_x(1,1) = f_y(1,1) = -\frac{1}{\sqrt{2}}.$$
By definition $f_x(1,1) = -1/\sqrt{2}$ is the slope of the curve of intersection of the hemisphere $z$ with the plane $y = 1$ at $x = 1$ (this is the geometrical definition of derivative we saw in the class). Similarly, the slope of the curve of intersection of the hemisphere $z$ with the plane $x = 1$ at $y = 1$ is $f_y(1,1) = -1/\sqrt{2}$. Therefore the equation of the tangent line to the hemisphere which passes by the point $(1,1,\sqrt{2})$ is of the form:
$$z = a + by,$$
with
$$b = f_y(1,1) = -1/\sqrt{2},$$
and $a = 3/\sqrt{2}$ being fixed by the condition that the line passes by the point $(1,1,\sqrt{2})$.

4.
$$f_x = \frac{y}{x(x+y)}, \qquad f_y = -\frac{1}{x+y},$$
$$f_{xx} = -\frac{1}{x^2} + \frac{1}{(x+y)^2}, \quad f_{yy} = \frac{1}{(x+y)^2}, \quad f_{xy} = f_{yx} = \frac{1}{(x+y)^2}.$$
For the function $g$ it is useful to write $g = e^{\ln(x^y)} = e^{y\ln(x)}$ in order to compute the derivatives:
$$g_x = x^{y-1}y, \qquad g_y = x^y\ln(x),$$
$$g_{xx} = y(y-1)x^{y-2}, \quad g_{yy} = x^y\ln^2(x), \quad g_{xy} = g_{yx} = x^{y-1}(1 + y\ln(x)).$$
$$r_x = e^y + ye^x, \qquad r_y = xe^y + e^x,$$
$$r_{xx} = ye^x, \quad r_{yy} = xe^y, \quad r_{xy} = r_{yx} = e^y + e^x.$$

5. Here the best way to proceed is to use the definitions of $f_{xy}$ and $f_{yx}$ given in page 16 and compute explicitly the limits. By doing that one obtains:
$$f_{xy}(0,0) = -1, \qquad f_{yx}(0,0) = 1.$$