Solutions to sheet 3

1. Here we have to use the chain rule, namely the equations

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}, \qquad \frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}. \tag{0.1}$$

For the change of variables $x = 2s + 3t$ and $y = 3s - 2t$ we have $\frac{\partial x}{\partial s} = -\frac{\partial y}{\partial t} = 2$, $\frac{\partial x}{\partial t} = \frac{\partial y}{\partial s} = 3$, and therefore we obtain $z_s = 2z_x + 3z_y$, and $z_t = 3z_x - 2z_y$. Using the chain rule again:

$$\begin{aligned}
\frac{\partial^2 z}{\partial s^2} &= \frac{\partial}{\partial s}\left(\frac{\partial z}{\partial s}\right) = \frac{\partial}{\partial s}(2z_x + 3z_y) = 2\frac{\partial z_x}{\partial s} + 3\frac{\partial z_y}{\partial s} \\
&= 2(2z_{xx} + 3z_{yx}) + 3(2z_{xy} + 3z_{yy}) = 4z_{xx} + 6z_{yx} + 6z_{xy} + 9z_{yy}, \\
\frac{\partial^2 z}{\partial t^2} &= \frac{\partial}{\partial t}\left(\frac{\partial z}{\partial t}\right) = \frac{\partial}{\partial t}(3z_x - 2z_y) = 3\frac{\partial z_x}{\partial t} - 2\frac{\partial z_y}{\partial t} \\
&= 3(3z_{xx} - 2z_{yx}) - 2(3z_{xy} - 2z_{yy}) = 9z_{xx} - 6z_{yx} - 6z_{xy} + 4z_{yy}, \\
\frac{\partial^2 z}{\partial s \partial t} &= \frac{\partial}{\partial s}\left(\frac{\partial z}{\partial t}\right) = \frac{\partial}{\partial s}(3z_x - 2z_y) = 3\frac{\partial z_x}{\partial s} - 2\frac{\partial z_y}{\partial s} \\
&= 3(2z_{xx} + 3z_{yx}) - 2(2z_{xy} + 3z_{yy}) = 6z_{xx} + 9z_{yx} - 4z_{xy} - 6z_{yy}.
\end{aligned}$$

2. For this problem $x = e^s\cos(t)$ and $y = e^s\sin(t)$, therefore:

$$\frac{\partial x}{\partial s} = \frac{\partial y}{\partial t} = e^s\cos(t), \quad \frac{\partial x}{\partial t} = -\frac{\partial y}{\partial s} = -e^s\sin(t).$$

The equations (0.1) become $f_s = e^s\cos(t)f_x + e^s\sin(t)f_y$, and $f_t = -e^s\sin(t)f_x + e^s\cos(t)f_y$. Using the chain rule again (as in the first problem) we obtain:

$$\begin{aligned}
\frac{\partial^2 f}{\partial s^2} &= \frac{\partial}{\partial s}\left(\frac{\partial f}{\partial s}\right) = \frac{\partial}{\partial s}\left(e^s\cos(t)f_x + e^s\sin(t)f_y\right) \\
&= e^s\cos(t)f_x + e^{2s}\cos(t)^2 f_{xx} + e^{2s}\cos(t)\sin(t)(f_{yx} + f_{xy}) + e^s\sin(t)f_y + e^{2s}\sin(t)^2 f_{yy}, \\
\frac{\partial^2 f}{\partial t^2} &= \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial t}\right) = \frac{\partial}{\partial t}\left(-e^s\sin(t)f_x + e^s\cos(t)f_y\right) \\
&= -e^s\cos(t)f_x + e^{2s}\sin(t)^2 f_{xx} - e^{2s}\cos(t)\sin(t)(f_{yx} + f_{xy}) - e^s\sin(t)f_y + e^{2s}\cos(t)^2 f_{yy}.
\end{aligned}$$

Therefore

$$\frac{\partial^2 f}{\partial s^2} + \frac{\partial^2 f}{\partial t^2} = e^{2s}(f_{xx} + f_{yy}) = (x^2 + y^2)(f_{xx} + f_{yy}).$$

3. In this problem we have a function $T(x, y, z, t)$ of 4 variables!

(a) Rate of change is equivalent to saying dT/dt. We use the chain rule for the change of variables $x = f(t)$, $y = g(t)$, $z = h(t)$ and $t = t$ and we obtain

$$\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{\partial z}\frac{dz}{dt} + \frac{\partial T}{\partial t}\frac{dt}{dt} = \frac{\partial T}{\partial x}f'(t) + \frac{\partial T}{\partial y}g'(t) + \frac{\partial T}{\partial z}h'(t) + \frac{\partial T}{\partial t}.$$

(b) For the function given we have

$$\frac{\partial T}{\partial x} = \frac{-200x}{(5+x^2+y^2)^2}\left(1+\sin\frac{\pi t}{12}\right), \quad \frac{\partial T}{\partial y} = \frac{-200y}{(5+x^2+y^2)^2}\left(1+\sin\frac{\pi t}{12}\right),$$

$$\frac{\partial T}{\partial z} = -40z, \qquad \frac{\partial T}{\partial t} = \frac{25\pi}{3(5+x^2+y^2)}\cos\frac{\pi t}{12}.$$

and

$$f'(t) = 1, \quad g'(t) = 2, \quad h'(t) = 1 - 2t^3.$$

Substituting these derivatives in the formula of part (a) and setting $t = 1$ you get $\frac{dT}{dt}\Big|_{t=1} = 9.9406...$

4. This problem is very similar to exercises 1 and 2. Again we need to use the chain rule to compute the first and second-order derivatives with respect to the new variables. We obtain

$$f_r = \cos\theta f_x + \sin\theta f_y, \quad f_\theta = -r\sin\theta f_x + r\cos\theta f_y.$$

For the second-order derivatives we have

$$\begin{aligned} f_{rr} &= \cos^2\theta f_{xx} + \sin^2\theta f_{yy} + 2\sin\theta\cos\theta f_{xy}, \\ f_{\theta\theta} &= -r\cos\theta f_x - r\sin\theta f_y + r^2\sin^2\theta f_{xx} + r^2\cos^2\theta f_{yy} - 2r^2\sin\theta\cos\theta f_{xy}, \end{aligned}$$

and therefore $f_{rr} + \frac{1}{r^2}f_{\theta\theta} + \frac{1}{r}f_r = f_{xx} + f_{yy}$.