Solutions to sheet 4

1.
$$du = u_x dx + u_y dy + u_z dz = \frac{y\left(y^2+z^2\right)}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}dx + \frac{x\left(x^2+z^2\right)}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}dy - \frac{xyz}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}dz$$

For $dx = 1/2, dy = 1/4$ and $dz = -1/4$ and $(x, y, z) = (1, 3, -2)$ we obtain

$$du = \frac{39}{28\sqrt{14}} + \frac{5}{56\sqrt{14}} - \frac{3}{28\sqrt{14}} = \frac{11}{8\sqrt{14}} = 0.3674...$$

2.
$$df = (-2x/(x^2+y^2-1)^2)dx + (-2y/(x^2+y^2-1)^2)dy.$$

Since $f(1,-1) = 1$, $f_x(1,-1) = -2 = -f_y(1,-1)$, $dx = 1.01 - 1 = 0.01$ and $dy = (-1.003+1) = -0.003$ we obtain $f(1.01, -1.003) \approx f(1,-1) + df = 1 - 2(0.01 + 0.003) = 0.974$.

3. We saw in the lecture that $d^2 f = f_{xx}dx^2 + f_{yy}dy^2 + 2f_{xy}dxdy$.
(a)
$$f_x = e^{x-y} + e^{y-x} = -f_y, \quad f_{xx} = f_{yy} = -f_{xy} = f(x,y).$$

Since $f(1,2) = -e + 1/e$, $f_x(1,2) = e + 1/e$, we get $d^2 f = 4(-e+1/e)(0.01)^2 = -0.000940161$
(b) Writing $g(x,y) = e^{x\log y + y\log x}$ we get

$$g_x = (\log y + y/x)g, \quad g_y = (\log x + x/y)g, \quad g_{xx} = (-y/x^2)g + (\log y + y/x)^2 g$$

$$g_{yy} = (-x/y^2)g + (\log x + x/y)^2 g, \qquad g_{xy} = (1/y + 1/x)g + (\log y + y/x)(\log x + x/y)g.$$

The values of the functions at $(1,2)$ are,

$$g(1,2) = 2, \quad g_x(1,2) = 4 + 2\log 2, \quad g_y(1,2) = 1, \quad g_{xx}(1,2) = (4 + 8\log 2 + 2(\log 2)^2),$$

$$g_{yy}(1,2) = 0, \quad g_{xy}(1,2) = 5 + \log 2,$$

which gives $d^2 g = (4 + 8\log 2 + 2(\log 2)^2)(0.01)^2 - (5 + \log 2)(0.01)^2 = 0.000481294$.

4. We define $G(x,y,z) = z^2 - 2xy^3 - \frac{xz}{y} = 0$. Then

$$\frac{\partial z}{\partial x} = -\frac{G_x}{G_z} = -\frac{-2y^3 - z/y}{2z - x/y} = \frac{2y^4 + z}{2zy - x}, \qquad \frac{\partial z}{\partial y} = -\frac{G_y}{G_z} = -\frac{-6xy^2 + xz/y^2}{2z - x/y} = \frac{6xy^4 - xz}{y(2zy - x)}.$$

For $(x,y) = (1,1)$ we have that $G(z,1,1) = z^2 - 2 - z = 0$. This equation has solutions $z = -1, 2$. The problem tells us to choose the positive solution, that is $z = 2$. For $x = y = 1$ and $z = 2$ we obtain $\frac{\partial z}{\partial x}\Big|_{(1,1)} = \frac{\partial z}{\partial y}\Big|_{(1,1)} = \frac{4}{3}$.

5. Here we can compute the derivatives in two ways: from the formulae for implicit functions,

$$\frac{\partial z}{\partial x} = -\frac{G_x}{G_z} = -\frac{x}{z}, \qquad \frac{\partial z}{\partial y} = -\frac{G_y}{G_z} = -\frac{y}{z}.$$

or by solving the equation $G(x,y,z) = 0$ for z, $z = \sqrt{4 - x^2 - y^2}$, and then computing the derivatives directly

$$\frac{\partial z}{\partial x} = -\frac{x}{\sqrt{4 - x^2 - y^2}} = -\frac{x}{z}, \qquad \frac{\partial z}{\partial y} = -\frac{y}{\sqrt{4 - x^2 - y^2}} = -\frac{y}{z}.$$

6. Here we have to use the Taylor expansion up to second order terms:

$$\begin{aligned} f(x,y) &= f(x_0,y_0) + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0) \\ &+ \frac{1}{2}f_{xx}(x_0,y_0)(x - x_0)^2 + \frac{1}{2}f_{yy}(x_0,y_0)(y - y_0)^2 + f_{xy}(x_0,y_0)(x - x_0)(y - y_0), \end{aligned}$$

where we assumed that $f_{xy} = f_{yx}$. For $(x_0, y_0) = (1, 2)$ and $f(x, y) = \sqrt{x^2 + y^3}$. First we compute the 1st and 2nd order derivatives:

$$f_x = \frac{x}{\sqrt{x^2+y^3}}, \quad f_y = \frac{3y^2}{2\sqrt{x^2+y^3}}, \quad f_{xx} = \frac{y^3}{(x^2+y^3)^{3/2}}, \quad f_{yy} = \frac{3(4x^2y+y^4)}{4(x^2+y^3)^{3/2}}, \quad f_{xy} = f_{yx} = -\frac{3xy^2}{2(x^2+y^3)^{3/2}},$$

$$f(1,2) = 3, \quad f_x(1,2) = \frac{1}{3}, \quad f_y(1,2) = 2, \quad f_{xx}(1,2) = \frac{8}{27}, \quad f_{yy}(1,2) = \frac{2}{3}, \quad f_{xy}(1,2) = f_{yx}(1,2) = -\frac{2}{9}.$$

Therefore, the Taylor expansion is

$$f(x,y) = \frac{1}{27}(-8 + 13x + 24y + 9y^2 + 4x^2 - 6xy).$$

The approximate value of the function at the point $(1.02, 1.97)$ is therefore

$$f(1.02, 1, 97) \approx \frac{1}{27}(-8 + 13(1.02) + 24(1.97) + 9(1.97)^2 + 4(1.02)^2 - 6(1.02)(1.97)) = 2.947159...$$

The exact value of the function at this point is $\sqrt{(1.02)^2 + (1.97)^3} = 2.947163...$ Therefore the Taylor approximation is extremely good in this case!

7. Here we have to consider the Taylor expansion around a point (x_0, y_0) up to 3rd order derivatives. We can obtain this expansion from the class notes,

$$\begin{aligned} f(x,y) &= f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \\ &+ \frac{1}{2}f_{xx}(x_0,y_0)(x-x_0)^2 + \frac{1}{2}f_{yy}(x_0,y_0)(y-y_0)^2 + f_{xy}(x_0,y_0)(x-x_0)(y-y_0) \\ &+ \frac{1}{3!}f_{xxx}(x_0,y_0)(x-x_0)^3 + \frac{1}{3!}f_{yyy}(x_0,y_0)(y-y_0)^3 \\ &+ \frac{3}{3!}f_{xxy}(x_0,y_0)(x-x_0)^2(y-y_0) + \frac{3}{3!}f_{yyx}(x_0,y_0)(y-y_0)^2(x-x_0), \end{aligned}$$

where we assumed $f_{xy} = f_{yx}$, $f_{xxy} = f_{yxx} = f_{xyx}$ and $f_{yyx} = f_{xyy} = f_{yxy}$. In our example we want to take $(x_0, y_0) = (2, 1)$ and $f(x, y) = (2 + x - 2y)^{-1}$. As before, we need to compute the derivatives

$$\begin{aligned} f_x &= -\frac{1}{(2+x-2y)^2} = -\frac{f_y}{2}, \qquad f_{xx} = \frac{2}{(2+x-2y)^3} = \frac{f_{yy}}{4} = -\frac{f_{xy}}{2}, \\ f_{xxx} &= -\frac{6}{(2+x-2y)^4} = -\frac{f_{xxy}}{2} = \frac{f_{yyx}}{4} = -\frac{f_{yyy}}{8}. \end{aligned}$$

At the point $(2, 1)$

$$\begin{aligned} f(2,1) &= \frac{1}{2}, \quad f_x(2,1) = -\frac{1}{4} = -\frac{f_y(2,1)}{2}, \qquad f_{xx}(2,1) = \frac{1}{4} = \frac{f_{yy}(2,1)}{4} = -\frac{f_{xy}(2,1)}{2}, \\ f_{xxx}(2,1) &= -\frac{3}{8} = -\frac{f_{xxy}(2,1)}{2} = \frac{f_{yyx}(2,1)}{4} = -\frac{f_{yyy}(2,1)}{8}. \end{aligned}$$

So, the Taylor expansion becomes

$$f(x,y) = \frac{1}{16}(8 - 4x + 2x^2 - x^3 + 8y - 8xy + 6x^2y + 8y^2 - 12xy^2 + 8y^3).$$

8. In this case we have to proceed as in problem 1. We compute the derivatives :

$$f_x = \frac{2x}{x^2+y^2}, \quad f_y = \frac{2y}{x^2+y^2}, \quad f_{xx} = \frac{-2\,(x-y)\,(x+y)}{(x^2+y^2)^2} = -f_{yy}, \quad f_{xy} = \frac{-4\,x\,y}{(x^2+y^2)^2}.$$

$$f(1,0) = 0, \quad f_x(1,0) = 2, \quad f_y(1,0) = 0, \quad f_{xx}(1,0) = -2 = -f_{yy}(1,0), \quad f_{xy}(1,0) = 0.$$

Therefore the Taylor expansion around the point $(1, 0)$ is:

$$f(x,y) = 2(x-1) - (x-1)^2 + y^2 = -3 + 4x - x^2 + y^2.$$