Solutions to sheet 5

1. We first find the points at which both f_x and f_y vanish:

$$f_x = 2x - 4, \qquad f_y = 4y + 4,$$

therefore $f_x = 0 \quad \Leftrightarrow \quad x = 2$ and $f_y = 0 \quad \Leftrightarrow \quad y = -1$, and the only point at which both derivatives vanish is $(2, -1)$. The 2nd order derivatives are: $f_{xx} = 2$, $f_{yy} = 4$ and $f_{xy} = 0$. Therefore

$$f_{xx}(2,-1) f_{yy}(2,-1) - f_{xy}(2,-1)^2 = 8 > 0,$$

and since $f_{xx}(2,-1) > 0$ the point $(2,-1)$ is a <u>minimum</u>.

2. As before we compute the 1st order partial derivatives:

$$f_x = 2e^{2x+3y}\left(8x^2 + x\,(8 - 6y) + 3\,(-1 + y)\,y\right), \qquad f_y = 3e^{2x+3y}\left(8x^2 - 2x\,(1 + 3y) + y\,(2 + 3y)\right).$$

They both vanish at the points $(0,0)$ and $(-1/4, -1/2)$. The 2nd order partial derivatives are:

$$f_{xx} = 4e^{2x+3y}\left(4 + 16x + 8x^2 - 6y - 6xy + 3y^2\right), \quad f_{yy} = 3e^{2x+3y}\left(2 - 12x + 24x^2 + 12y - 18xy + 9y^2\right),$$

and

$$f_{xy} = f_{yx} = 6e^{2x+3y}\left(-1 + 6x + 8x^2 - y - 6xy + 3y^2\right).$$

We have

$$f_{xx}(0,0) = 16, \qquad f_{yy}(0,0) = 6, \qquad f_{xy}(0,0) = -6,$$

and

$$f_{xx}(-1/4,-1/2) = \frac{14}{e^2}, \qquad f_{yy}(-1/4,-1/2) = \frac{3}{2e^2}, \qquad f_{xy}(-1/4,-1/2) = -\frac{9}{e^2}.$$

With this we get that the point $(0,0)$ is a <u>minimum</u> and the point $(-1/4,-1/2)$ is a <u>saddle point</u>.

3. We start by finding the zeros of the 1st order partial derivatives:

$$f_x = 1 - \frac{1}{x^2 y} = \frac{x^2 y - 1}{x^2 y}, \qquad f_y = 8 - \frac{1}{xy^2} = \frac{8y^2 x - 1}{y^2 x}.$$

$$f_x = 0 \quad \Leftrightarrow \quad x^2 y = 1 \quad \text{and} \quad f_y = 0 \quad \Leftrightarrow \quad 8y^2 x = 1.$$

The two conditions are satisfied only for $x = 2$ and $y = 1/4$. Therefore the only candidate to be a minimum of the function is the point $(2, \frac{1}{4})$. We compute the 2nd order partial derivatives:

$$f_{xx} = \frac{2}{x^3 y}, \qquad f_{yy} = \frac{2}{xy^3}, \qquad f_{xy} = \frac{1}{x^2 y^2}.$$

Therefore

$$f_{xx}(2,1/4) = 1 > 0, \qquad f_{yy}(2,1/4) = 64, \qquad f_{xy}(2,1/4) = 4.$$

and the point $(2,1/4)$ is a <u>minimum</u> of the function. Hence the minimum value of the function in the first quadrant is $f(2,1/4) = 6$.

4. Defining $f(x,y) = x^3 y^5$ and $g(x,y) = x + y - 8$, the set of equations to be solved are

$$3x^2 y^5 + \lambda = 0, \qquad 5x^3 y^4 + \lambda = 0, \qquad x + y - 8 = 0.$$

The first two equations are solved by:

$$x = 0 \qquad \text{or} \qquad y = 0 \qquad \text{or} \qquad y = \frac{5x}{3}.$$

Putting each of these conditions into the last equation we obtain the following solutions:

$$x = 0, \quad y = 8 \quad \text{and} \quad \lambda = 0,$$

$$x = 8, \quad y = 0 \quad \text{and} \quad \lambda = 0,$$

$$x = 3, \quad y = 5 \quad \text{and} \quad \lambda = -3^3 5^5.$$

We compute $f(0,8) = f(8,0) = 0$ and $f(3,5) = 3^3 5^5$. The value of $f(x,y)$ is maximum at the point $(3,5)$, therefore $x = 3, y = 5$ is the solution to the problem.

5. In this case we have to find both the maximum and minimum values of $f(x, y, z)$. The constraint is given by the function $g(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$. Therefore, we have to solve the following system of equations:

$$1 + \lambda 2x = 0, \qquad 1 + \lambda 2y = 0, \qquad -1 + \lambda 2z = 0, \qquad x^2 + y^2 + z^2 - 1 = 0.$$

From the 1st three equations we get:

$$\lambda = -\frac{1}{2x} = -\frac{1}{2y} = \frac{1}{2z},$$

therefore the solution must have $x = y = -z$. Putting this into the last equation we get the condition:

$$3x^2 = 1 \qquad \Leftrightarrow \qquad x = \pm\sqrt{\frac{1}{3}}.$$

Therefore we get two solutions, namely: $x = y = -z = \sqrt{\frac{1}{3}}$ and $\lambda = -\frac{\sqrt{3}}{2}$, and $x = y = -z = -\sqrt{\frac{1}{3}}$ and $\lambda = \frac{\sqrt{3}}{2}$. One of these solutions will be the maximum and the other the minimum. We compute the values of $f(x, y, z)$ at the two points: $f(\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}, -\sqrt{\frac{1}{3}}) = \sqrt{3}$, and $f(-\sqrt{\frac{1}{3}}, -\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}) = -\sqrt{3}$. Therefore the first point maximizes the function and the second one minimizes it.

6. The equations we need to solve are:

$$2x + \lambda 2xy = 0, \qquad 2y + \lambda x^2 = 0, \qquad x^2y - 16 = 0.$$

The first equation gives:

$$2x(1 + \lambda y) = 0,$$

and has solutions $x = 0$ or $y = -1/\lambda$. If we put $x = 0$ into the second equation we get that $y = 0$ too and this is incompatible with the last equation. Therefore this is not a valid solution. The other possibility was $y = -1/\lambda$ and in this case the second equation becomes:

$$-2 + \lambda^2 x^2 = 0 \quad \Leftrightarrow \quad x = \pm\frac{\sqrt{2}}{\lambda}.$$

Putting this into the last equation we obtain $\lambda^3 = -1/8$, that is $\lambda = -1/2$. Putting all these results together we get again two solutions:

$$x = \frac{\sqrt{2}}{\lambda} = -2\sqrt{2}, \quad y = 2, \quad \lambda = -1/2,$$

and

$$x = -\frac{\sqrt{2}}{\lambda} = 2\sqrt{2}, \quad y = 2, \quad \lambda = -1/2.$$

The problems asks us to find the solution that produces the minimum value of $f(x, y)$ so we compute $f(2\sqrt{2}, 2) = 12$, and $f(-2\sqrt{2}, 2) = 12$. Both points give the same value of $f(x, y)$ so we conclude that there are two solutions that minimize the value of $f(x, y)$.

7. Here it is crucial to realize that although a point in three dimensional space is involved, the problem is two-dimensional, as we are looking for points (x, y) that lie on the given curve, which lives itself in the xy-plane. The square distance is $f(x, y) = x^2 + y^2 + 1$, and the equations to solve are,

$$y^2 + x^2 + 4xy - 4 = 0, \quad 2x + \lambda(2x + 4y) = 0, \quad 2y + \lambda(2y + 4x) = 0.$$

Subtracting the last two equations we get $2(x-y)+\lambda(2(x-y)+4(y-x)) = 2(x-y)(1-\lambda) = 0$ which admits solutions $x = y$ or $\lambda = 1$. Putting $x = y$ into the constraint we obtain $6x^2 - 4 = 0$ which has solutions $x = y = \pm\sqrt{2/3}$. The value of λ is the same for both solutions $\lambda = -1/6$. If we take the solution $\lambda = 1$ and put it into the second equation we get $x = -y$. Plugging this into the constraint we obtain an equation that has not real solutions. Therefore we obtain two solutions $(\sqrt{2/3}, \sqrt{2/3})$ and $(-\sqrt{2/3}, -\sqrt{2/3})$ and the two points are at the same distance from the curve, which is minimal.