Solutions to coursework 1

1. **(a)** The definitions of f_x and f_y in terms of limits are:

$$f_x(x,y) = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}, \qquad f_y(x,y) = \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}.$$

We need to carry out these limits now for $(x,y)=(0,0)$, therefore we have

$$f_x(0,0) = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}, \qquad f_y(0,0) = \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}.$$

From the function's definition we see that $f(h,0)=f(0,h)=f(0,0)=0$, therefore the limits become

$$f_x(0,0)=f_y(0,0)=\lim_{h\to 0}\frac{0}{h}=\lim_{h\to 0}\frac{0}{1}=0,$$

where in the last equality I have used L'Hôpital's rule. Therefore the answer to this question is $f_x(0,0)=f_y(0,0)=0$.
Feedback: Many students got this part wrong for different reasons. Many failed to realize that $f(0,0)=0$, even though it is given by the second line of the equation that defines $f(x,y)$. Many of you tried to compute $f(0,0)$ by substituting $x=y=0$ into the first line of the equation. If you do this you get 0/0 which is <u>indefinite</u>. Having got that, many students just said 0/0=0 which is wrong! There were other sorts of mistakes, such as not writing the definition in terms of limits correctly, forgetting writing the limit sign at all or just ignoring $f(0,0)$ without ever saying that it is zero or computing the limits wrongly.
Marking: 5 points for each derivative with 2.5 points for using the correct formula and 2.5 points for carrying out the limit correctly.

(b) For this question we can just use the definition of the function for $(x,y)\neq(0,0)$ and differentiate with respect to x and y in the usual way. We find,

$$f_x(x,y)=\frac{\partial}{\partial x}\left(\frac{xy(x^2-y^2)}{x^2+y^2}\right)=\frac{y(3x^2-y^2)(x^2+y^2)-xy(x^2-y^2)2x}{(x^2+y^2)^2}$$

$$=\frac{3yx^4+3y^3x^2-y^3x^2-y^5-2x^4y+2x^2y^3}{(x^2+y^2)^2}=\frac{y(x^4-y^4+4y^2x^2)}{(x^2+y^2)^2}.$$

Similarly for $f_y(x,y)$ we obtain

$$f_y(x,y)=\frac{\partial}{\partial y}\left(\frac{xy(x^2-y^2)}{x^2+y^2}\right)=\frac{x(x^2-3y^2)(x^2+y^2)-xy(x^2-y^2)2y}{(x^2+y^2)^2}$$

$$=\frac{x^5+x^3y^2-3y^2x^3-3y^4x-2x^3y^2+2y^4x)}{(x^2+y^2)^2}=\frac{x(x^4-y^4-4x^2y^2)}{(x^2+y^2)^2}.$$

Feedback: This was mostly OK, although there was a group of students who got zero for this question because having computed the derivative wrongly (or at least forgotten some terms) went on to get the correct result! This could only happen to so many students (who made the exact same mistake) because they copied from each other without thinking what they were copying!
Marking: 5 points for each derivative out of which 1.5 points will be subtracted if the expression is not given in a simplified form.

(c) Now we need to use the Existence theorem for the limit that we saw in the class. This means that we need to prove that if we assume that $0 < \sqrt{x^2+y^2} < \delta(\epsilon)$ then it is possible to find ϵ such that

$$\left|\frac{xy(x^2-y^2)}{x^2+y^2}\right| < \epsilon$$

The strategy is the same as for the examples we did in the lecture. We consider the absolute value above and use the inequalities $|x| \leq \sqrt{x^2+y^2}$ and $|y| \leq \sqrt{x^2+y^2}$ to write

$$\left|\frac{xy(x^2-y^2)}{x^2+y^2}\right| \leq \left|\frac{(x^2+y^2)(x^2-y^2)}{x^2+y^2}\right| = |x^2-y^2| \leq |x^2+y^2-y^2| = x^2 \leq x^2+y^2 < \delta(\epsilon)^2.$$

Therefore, if we choose $\epsilon = \delta(\epsilon)^2$ we can always guarantee that the theorem is satisfied.

Feedback: Lots of people got this question wrong. First of all, because the question says "proof" it means that you need to do it using the theorem. This is the only way in which you can conclusively proof that the limit exists and is zero. Some students tried a couple of trajectories, got zero and then said this was a proof. But it is not! To have a proof you would have to try every possible trajectory that contains (0,0) and check that the limit is zero. Since there are infinitely many such trajectories, you can not do that, so instead you use a theorem that states precisely when the limit exists.

Those students who try to use the theorem, often used it wrongly. The trick was similar to the examples I did in the class, but lots of people (who probably copied from each other!) made one particular mistake. The mistake was to say that $|xy(x^2-y^2)| = |x^3y| - |xy^3|$ which is wrong! In general, it is NOT true to say that $|A-B| = |A| - |B|$ as you can easily see if you take $A = 2$ and $B = 6$, for example! This mistake then lead to the whole question being wrong.

Marking: 3 points for specializing the theorem correctly to the present case, 7 points for a correct proof.

2. **(a)** Both for parts (a) and (b) we need to compute all first and second order derivatives. They are:

$$f_x = 2e^{2x+3y}\sin(x^2-y^4)+2xe^{2x+3y}\cos(x^2-y^4) = 2e^{2x+3y}(\sin(x^2-y^4)+x\cos(x^2-y^4)),$$

$$f_y = 3e^{2x+3y}\sin(x^2-y^4)-4y^3e^{2x+3y}\cos(x^2-y^4) = e^{2x+3y}(3\sin(x^2-y^4)-4y^3\cos(x^2-y^4)),$$

$$\begin{aligned} f_{xx} &= 4e^{2x+3y}(\sin(x^2-y^4)+x\cos(x^2-y^4))+2e^{2x+3y}(2x\cos(x^2-y^4)+\cos(x^2-y^4)-2x^2\sin(x^2-y^4)) \\ &= 4e^{2x+3y}(1-x^2)\sin(x^2-y^4)+2e^{2x+3y}(4x+1)\cos(x^2-y^4) \end{aligned}$$

$$\begin{aligned} f_{yy} &= 3e^{2x+3y}(3\sin(x^2-y^4)-4y^3\cos(x^2-y^4)) \\ &\quad +e^{2x+3y}(-12y^3\cos(x^2-y^4)-12y^2\cos(x^2-y^4)-16y^6\sin(x^2-y^4)) \\ &= e^{2x+3y}(9-16y^6)\sin(x^2-y^4)-12y^2(2y+1)e^{2x+3y}\cos(x^2-y^4), \\ f_{xy} &= f_{yx} = 6e^{2x+3y}(\sin(x^2-y^4)+x\cos(x^2-y^4)) \\ &\quad +2e^{2x+3y}(-4y^3\cos(x^2-y^4)+4y^3x\sin(x^2-y^4)) \\ &= 2e^{2x+3y}(3+4y^3x)\sin(x^2-y^4)+2e^{2x+3y}(3x-4y^3)\cos(x^2-y^4). \end{aligned}$$

We need to evaluate the function and the derivatives at the point (1,1), which gives:

$$f(1,1) = 0, \quad f_x(1,1) = 2e^5, \quad f_y(1,1) = -4e^5,$$

$$f_{xx}(1,1) = 10e^5, \quad f_{yy}(1,1) = -36e^5, \quad f_{xy}(1,1) = -2e^5.$$

The Taylor expansion at second order is:

$$f(x,y) \approx f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$+\frac{1}{2}f_{xx}(x_0,y_0)(x-x_0)^2 + \frac{1}{2}f_{xx}(x_0,y_0)(y-y_0)^2 + f_{xy}(x_0,y_0)(x-x_0)(y-y_0).$$

Substituting the values for this case we get:

$$f(x,y) \approx 2e^5(x-1) - 4e^5(y-1) + 5e^5(x-1)^2 - 18e^5(y-1)^2 - 2e^5(x-1)(y-1)$$

$$= e^5(-13 - 6x + 5x^2 + 34y - 2xy - 18y^2).$$

Feedback: Most students got this right, although some failed to simplify the final formula completely and lost 1 point for that reason.
Marking: 1.5 points for each derivative (7.5 points), 3 points for getting the correct values of f and its derivatives at (1,1), 1.5 points for using the correct Taylor formula, 3 points for getting the final result in the most simplified form.

(b) For this part we just need to compute the second differential. We saw the formula in the class:

$$d^2f = f_{xx}d^2x + f_{yy}d^2y + 2f_{xy}dxdy,$$

In our case we need the values of f_{xx}, f_{yy}, f_{xy} at (1,1) computed above and the differentials,

$$dx = 0.1, \qquad dy = -0.02,$$

are the distance between the original coordinates (1,1) and the point at which we are evaluating the differential (1.1, 0.98). Therefore

$$d^2f = 10e^5(0.1)^2 - 36e^5(-0.02)^2 + 2(-2)e^5(0.1)(-0.02) = 13.8915...$$

Feedback: For some reason half of the class got this question wrong and got zero points for it. The mistake was that they computed something completely different, which I did not ask for at all! Many people compute the value of $f(x,y)$ at the point $(1.1, 0.98)$, but that has nothing to do with the second order differential! Some of the people who actually computed the differential got the signs of dx and dy wrong.
Marking: 1.5 points for using the correct formula for the differential, 2 points for identifying dx, dy correctly and 1.5 points for getting the correct value of d^2f.

3. The chain rule gives,

$$u_x = u_r r_x + u_s s_x = u_r + u_s, \quad u_y = u_r r_y + u_s s_y = u_r - u_s,$$

where we used $r_x = s_x = r_y = -s_y = 1$. The second derivatives u_{xx} and u_{yy} are:

$$u_{xx} = \frac{\partial}{\partial x}(u_x) = \frac{\partial u_r}{\partial x} + \frac{\partial u_s}{\partial x} = \underbrace{u_{rr} + u_{rs} + u_{sr} + u_{ss}}_{\text{using the chain rule above}},$$

$$u_{yy} = \frac{\partial}{\partial y}(u_y) = \frac{\partial u_r}{\partial y} - \frac{\partial u_s}{\partial y} = \underbrace{u_{rr} - u_{rs} - u_{sr} + u_{ss}}_{\text{using the chain rule above}},$$

From this it follows that

$$u_{xx} - u_{yy} = 4u_{rs}, \qquad \text{for} \quad u_{rs} = u_{sr}.$$

Therefore, if $u_{xx} - u_{yy} = 0$, then, in the new variables $u_{rs} = 0$.
Feedback: Most students got this question right and there are several ways of doing it. There were quite a few students that made mistakes which were obviously due to copying other people's work without thinking! They have been penalized for that. One particular mistake of this kind was for some people to write that $\frac{\partial}{\partial r}(\frac{\partial u}{\partial s}) = \frac{\partial u}{\partial r}\frac{\partial u}{\partial s}$. This is completely wrong (as you can easily check if you take the function $u = sr$!) The people who made that mistake went on to conclude that $\frac{\partial u}{\partial r}\frac{\partial u}{\partial s} = u_{rs}$ which is wrong, hence managing to get the final answer right! They lost half of the points awarded to the question.
Marking: 1.5 points for u_x, 1.5 points for u_y, 3 points for u_{xx}, 3 points for u_{yy} and 1 point for getting the final equation.

4. **(a)** Since we have 3 new variables, the chain rule generalizes as:

$$f_u = f_x x_u + f_y y_u, \quad f_v = f_x x_v + f_y y_v, \quad f_w = f_x x_w + f_y y_w.$$

In this case $x_u = \cos v$, $x_v = -(u+w)\sin v$, $x_w = \cos v$ and $y_u = \sin v$, $y_v = (u-w)\cos v$, $y_w = -\sin v$. So the answer to part (a) is:

$$f_u = \cos v f_x + \sin v f_y, \quad f_v = -(u+w)\sin v f_x + (u-w)\cos v f_y, \quad f_w = \cos v f_x - \sin v f_y.$$

Feedback: most students got this part right.

Marking: 4 points for writing the right general formulae, 3 points for the right x,y derivatives, 3 points for the correct final equations.

(b) If $f(x,y) = x^2 + y^2$, in terms of the new variables it becomes,

$$f(u,v,w) = (u+w)^2\cos^2 v + (u-w)^2\sin^2 v.$$

The derivative f_v is then,

$$f_v = (u+w)^2(-2)\cos v\sin v + (u-w)^2 2\sin v\cos v = \left(-(u+w)^2 + (u-w)^2\right)\sin(2v) = -4uw\sin(2v).$$

From this we can get the two derivatives f_{uv} and f_{vw}.

$$f_{uv} = f_{vu} = \frac{\partial f_v}{\partial u} = -4w\sin(2v),$$

and

$$f_{vw} = f_{wv} = \frac{\partial f_v}{\partial w} = -4u\sin(2v).$$

Feedback: many students got this part wrong, once again due largely to generalized copying. The easiest way of doing the question was to substitute x and y in the function by their expressions in terms of u, v, w. Many students took the formulae from the first part of the problem and substituted $f_x = 2x$ and $f_y = 2y$. This was ok, but because they ended up with an expression that mixed the variables x, y with the variables u, v, w they went on to compute the second derivatives wrongly. In particular they used that $dx/du = 0$, which is not true, since x is a function of u as given by the problem.
Marking: 2 points for $f(u, v, w)$, 3 points for f_v, 2.5 points for f_{uv} and 2.5 points for f_{wv}.

(a) We define the function

$$\Phi(x, y, z) = 3yz^2 - e^{4x}\cos(4z) - 3y^2 - 4 = 0,$$

and compute the derivatives

$$\phi_x = -4e^{4x}\cos(4z), \quad \Phi_y = 3z^2 - 6y, \quad \Phi_z = 6yz + 4e^{4x}\sin(4z).$$

Therefore,

$$\frac{\partial z}{\partial x} = -\frac{\Phi_x}{\Phi_z} = \frac{4e^{4x}\cos(4z)}{6yz + 4e^{4x}\sin(4z)},$$

$$\frac{\partial z}{\partial y} = -\frac{\Phi_y}{\Phi_z} = -\frac{3z^2 - 6y}{6yz + 4e^{4x}\sin(4z)}.$$

Feedback: most people got this question right. Some students computed the derivatives by differentiating the whole equation and others used the new method seen in the class. Since the problem did not specify a particular method, both were ok.
Marking: 6 points for the Φ derivatives, 2 points for z_x and 2 points for z_y.

(b) The value of z at $(x, y) = (1/2, 0)$ is obtained as:

$$-e^2\cos(4z) = 4 \quad \Leftrightarrow \quad \cos(4z) = -4e^{-2} \quad \Leftrightarrow \quad z = \frac{1}{4}\cos^{-1}(-4e^{-2}) = 0.535707 \quad \text{in radians}$$

$$= 30.6937 \quad \text{in degrees}$$

Substituting in the derivatives we obtain

$$\left.\frac{\partial z}{\partial x}\right|_{(1/2,0)} = \frac{-16}{4e^2\sin(4 * 0.535707)} \approx -0.643838,$$

$$\left.\frac{\partial z}{\partial y}\right|_{(1/2,0)} = -\frac{3(0.535707)^2}{4e^2\sin(4 * 0.535707)} \approx -0.138577 = -113.731 \quad \text{in degrees.}$$

Feedback: this question was also largely ok.
Marking: 3 points for z, 3.5 points for z_x and 3.5 points for z_y.