Solutions to coursework 2

1. For the function
$$f(x,y) = y^3 + yx^2 - 6x^2 - 6y^2 + 9y,$$
the first order partial derivatives are
$$\begin{aligned} f_x &= 2xy - 12x, \\ f_y &= 3y^2 + x^2 - 12y + 9. \end{aligned}$$
The first thing we have to do is finding the points at which these derivatives vanish
$$f_x = 0 \quad \Rightarrow \quad 2x(y-6) = 0 \quad \Rightarrow x = 0 \quad \text{or} \quad y = 6.$$
For $x = 0$ (which is one of the solutions of the previous equation) f_y will vanish if
$$f_y(0,y) = 0 = 3y^2 - 12y + 9 = 0 \;\Rightarrow\; y = \frac{12 \pm 6}{6} = 3, 1,$$
and for $y = 6$ (which is the other solution of $f_x = 0$) we would obtain
$$f_y(x,6) = 0 = x^2 + 45 \quad \Rightarrow \quad x = \pm i\sqrt{45}.$$
The last solution is not a real number and therefore it is not valid (since we are dealing with real functions of real variables!).

Putting all these solutions together we have that the points $(0,1)$ and $(0,3)$ are the only candidates to be stationary points of $f(x,y)$.

The next step is to compute the second order partial derivatives
$$\begin{aligned} A &= f_{xx} = 2y - 12, \\ B &= f_{xy} = f_{yx} = 2x, \\ C &= f_{yy} = 6y - 12, \end{aligned}$$
therefore
$$AC - B^2 = (2y-12)(6y-12) - 4x^2,$$
and we have to study the sign of this quantity in order to classify the stationary points of the function:

The point $(0,1)$: At this point
$$\begin{aligned} AC - B^2 &= (2-12)(6-12) = 60 > 0, \\ A &= 2 - 12 = -10 < 0, \end{aligned}$$
therefore this point is a **maximum**.

The point $(0,3)$: At this point
$$AC - B^2 = (6-12)(18-12) = -36 < 0,$$
therefore this point is a **saddle point**.

Feedback: Most people got this question right. Some failed to notice that there where complex solutions which are not valid, but still should be indicated.
Marking: 2 points for f_x and f_y, 2 points for the zeros of f_x, f_y, 1 point for the complex solutions, 2 points for the 2nd derivatives and 3 points for identifying $(0,1)$ as a maximum and 3 points for identifying $(0,3)$ as a saddle point.

2. The function that we want to maximize and minimize is the distance between a point (x, y, z) on the sphere and the point $(0, 0, 1)$ which lies on the z-axis. The distance function would be given by:

$$d(x, y, z) = \sqrt{x^2 + y^2 + (z-1)^2}.$$

Since the derivatives of this function are rather complicated, it is better to consider the square of the distance instead. This is correct because both the distance and its square will be maximal or minimal at the same points! So, let us consider the function

$$f(x, y, z) = x^2 + y^2 + (z-1)^2.$$

The constrain in this case is given by the fact that (x, y, z) must lie on the sphere:

$$\Phi(x, y, z) = (x-2)^2 + (y+1)^2 + (z-1)^2 - 4 = 0.$$

According to the method of Lagrange multipliers, we need to solve the following set of equations:

$$\begin{aligned} f_x + \lambda \Phi_x &= 0 \quad \Rightarrow \quad 2x + \lambda 2(x-2) = 0 & (0.1) \\ f_y + \lambda \Phi_y &= 0 \quad \Rightarrow \quad 2y + \lambda 2(y+1) = 0 & (0.2) \\ f_z + \lambda \Phi_z &= 0 \quad \Rightarrow \quad 2(z-1) + \lambda 2(z-1) = 0 & (0.3) \\ (x-2)^2 + (y+1)^2 + (z-1)^2 - 4 &= 0 & (0.4) \end{aligned}$$

In this case, probably the easiest way to start is to solve first equation (0.3). It clearly has two solutions which are:

$$z = 1 \quad \text{or} \quad \lambda = -1.$$

However, substituting the solution $\lambda = -1$ into equation (0.1) we find for instance that:

$$2x - 2(x-2) = 0 \quad \Rightarrow \quad 4 = 0!!$$

This of course not possible, therefore the solution $\lambda = -1$ is not consistence with the remaining equations and this means that the only solution to equation (0.3) is $z = 1$.

The two first equations can be re-written as:

$$\frac{x}{x-2} = \frac{y}{y+1} = -\lambda,$$

by solving each equation for λ. This implies that:

$$x(y+1) = y(x-2) \quad \Rightarrow \quad xy + x = yx - 2y \quad \Rightarrow \quad x = -2y.$$

Substituting this result, to together with $z = 1$ into equation (0.4) we find an equation for y which we can solve:

$$(-2y-2)^2 + (y+1)^2 - 4 = 0 \quad \Rightarrow \quad 5(y+1)^2 - 4 = 0 \quad \Rightarrow \quad y+1 = \pm\sqrt{\frac{4}{5}} \quad \Rightarrow \quad y = -1 \pm \frac{2}{\sqrt{5}}.$$

We therefore have two solutions for y which correspond to two solutions for x (since $x = -2y$) given by $x = 2 \mp \frac{4}{\sqrt{5}}$. Therefore we find two points on the sphere which are at maximal or minimal distance from $(0, 0, 1)$ and they are:

$$(x, y, z) = \left(2 - \frac{4}{\sqrt{5}}, -1 + \frac{2}{\sqrt{5}}, 1\right) \quad \text{and} \quad \left(2 + \frac{4}{\sqrt{5}}, -1 - \frac{2}{\sqrt{5}}, 1\right).$$

We still need to find the corresponding values of the Lagrange multiplier. For this we can just substitute the two different values of x for each point onto equation (0.1). This gives:

$$-\lambda = \frac{x}{x-2} = \frac{2 \mp \frac{4}{\sqrt{5}}}{\mp \frac{4}{\sqrt{5}}} = \mp \frac{\sqrt{5}}{2} + 1,$$

where the minus sign corresponds to the first point and the plus sign to the second point.

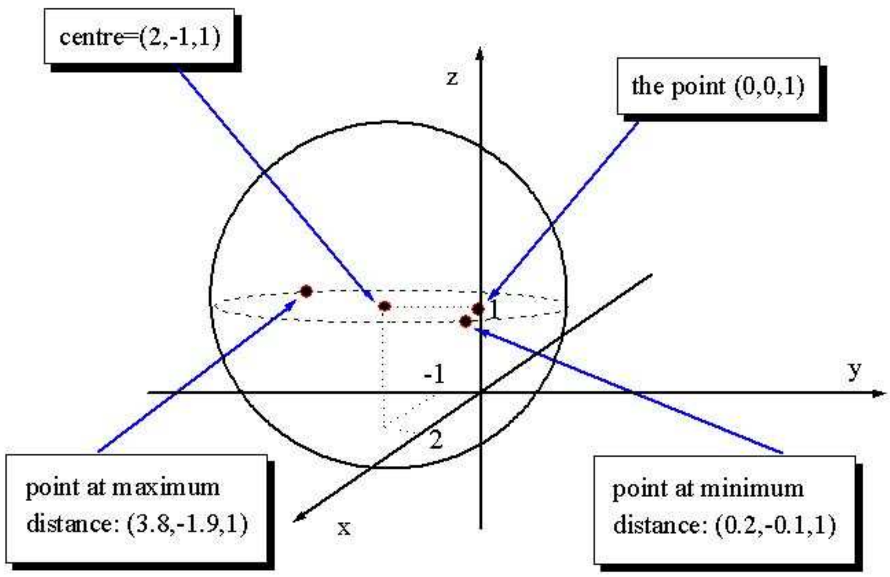
Now the only thing left to do is to find out which one of these points is a t maximum distance and which one at minimum distance. For this we just need to substitute the two solutions into the function $d(x, y, z)$ that we found above. We find:

$$d\left(2 - \frac{4}{\sqrt{5}}, -1 + \frac{2}{\sqrt{5}}, 1\right) = \sqrt{\left(2 - \frac{4}{\sqrt{5}}\right)^2 + \left(-1 + \frac{2}{\sqrt{5}}\right)^2} = \sqrt{9 - 4\sqrt{5}} = 0.236068...$$

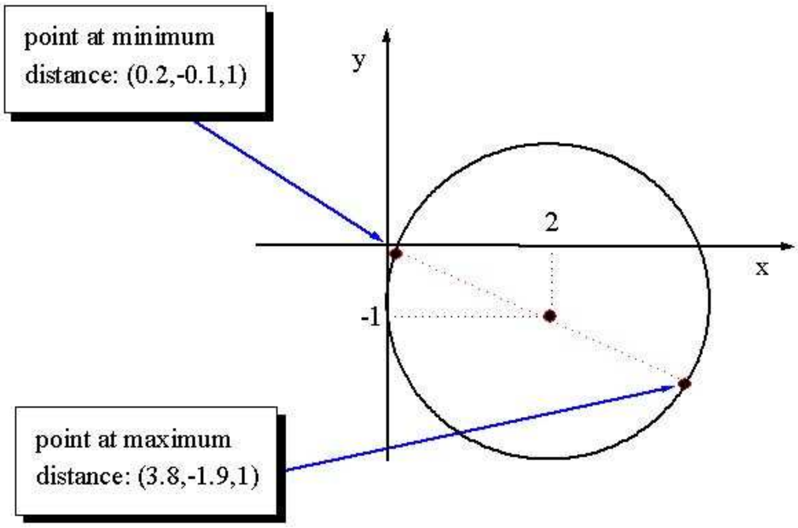
and

$$d\left(2 + \frac{4}{\sqrt{5}}, -1 - \frac{2}{\sqrt{5}}, 1\right) = \sqrt{\left(2 + \frac{4}{\sqrt{5}}\right)^2 + \left(-1 - \frac{2}{\sqrt{5}}\right)^2} = \sqrt{9 + 4\sqrt{5}} = 4.23607...$$

Clearly the distance is largest for the second point and therefore this provides the maximum whereas the first point is the one at minimum distance. The figure below shows the location of the points.



In fact the points can be seen better if we look instead at the intersection of the sphere with the plane $z = 1$ below (since the two points have $z = 1$, as well as the centre of the sphere).



From this picture we actually see that the two points are connected by a line that includes the centre of the sphere and the origin of coordinates (which in our picture corresponds to the point (0,0,1)) and that the largest distance must be equal to the shortest distance plus the diameter of the sphere (4), which is precisely what we got!

Feedback: Most people got part of the question right, although many forgot finding the values of λ or saying which point is the maximum and which one is the minimum. The most common error was to ignore the solution $\lambda = -1$ for which I subtracted one point.
Marking: 2 points for identifying the right functions f and Φ. 3 points for the equations, 2 points for the solution for z and $\lambda = -1$. 2 points for the solution $x = -2y$. 2 points for the values of x and y. 2 points for writing down the coordinates of the two solutions. 2 points for the Lagrange multipliers and 3 points for the distances in the most simplified form.

3. The Jacobian determinant is easy to obtain since we can re-write $x = (u+v)/2$ and $y = (v-u)/2$.

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}. \tag{0.5}$$

This means that $dxdy = \frac{1}{2} dudv$. The function $e^{\frac{x-y}{x+y}} = e^{\frac{u}{v}}$, therefore the integrand in the new coordinates is:

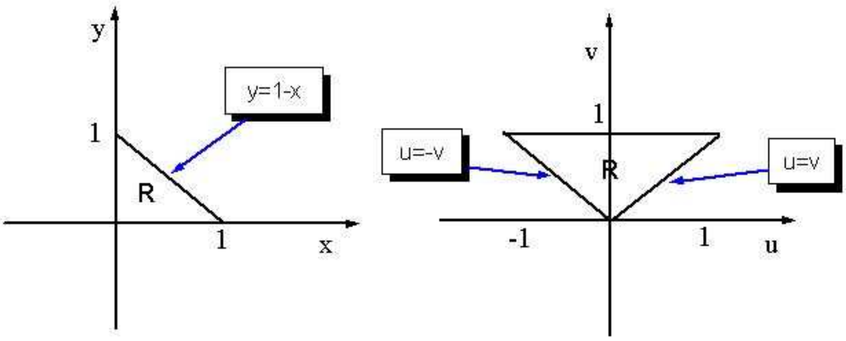
$$e^{\frac{x-y}{x+y}} dxdy = \frac{1}{2} e^{\frac{u}{v}} dudv.$$

The integration region, as described by the problem is the triangle with vertices $(0,1)$, $(0,0)$ and $(1,0)$. In the new coordinates (u,v) these points become:

$$(x,y) = (0,0) \quad \Rightarrow \quad (u,v) = (0,0)$$

$$(x, y) = (0, 1) \quad \Rightarrow \quad (u, v) = (-1, 1)$$
$$(x, y) = (1, 0) \quad \Rightarrow \quad (u, v) = (1, 1)$$

Therefore, the integration region in the new variables is also a triangle, albeit of a different shape (see figure below).

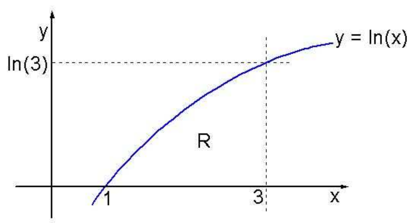


From the picture one realizes that it is easier to do the integral in u first and the integral in v last (it is very hard to integrate the function $e^{u/v}$ with respect to v).

$$I = \frac{1}{2}\int_{v=0}^{v=1} dv \int_{u=-v}^{u=v} e^{u/v} du = \frac{1}{2}\int_{v=0}^{v=1} dv [v e^{u/v}]_{u=-v}^{u=v} = \frac{e - e^{-1}}{2}\int_{v=0}^{v=1} v dv = \frac{e - e^{-1}}{2}[v^2/2]_0^1 = \frac{e - e^{-1}}{4}.$$

Feedback: Most students got the Jacobian wrong! The rest of the problem was generally ok.
Marking: 3 points for the Jacobian. 2 points for writing down the integrand in the new coordinates. 3 points for identifying the vertices of the new triangle. 6 points for the sketches. 6 points for doing the integral.

4. The integration region is



and therefore, reversing the order of integration we obtain

$$I = \int_{y=0}^{y=\ln(3)} \int_{x=e^y}^{x=3} x dx dy.$$

The x-integral gives

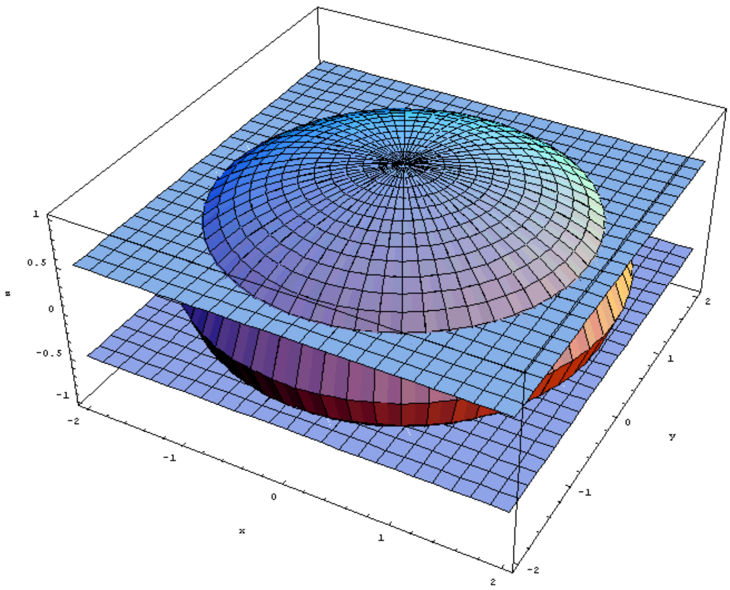
$$\int_{x=e^y}^{x=3} x dx = \left[\frac{x^2}{2}\right]_{x=e^y}^{x=3} = \frac{9-e^{2y}}{2}.$$

We obtain

$$I = \frac{1}{2}\int_{y=0}^{y=\ln(3)} (9-e^{2y}) dy = \frac{1}{2}\left[9y - \frac{e^{2y}}{2}\right]_{y=0}^{y=\ln(3)} = \frac{1}{2}\left(9\ln(3) - \frac{e^{2\ln(3)}}{2} + \frac{1}{2}\right) = \frac{1}{2}(9\ln(3)-4).$$

Feedback: This was mostly ok although few students had a wrong drawing for the integration region, some had the wrong integration limits when changing the order of integration and a few others did not change the order of integration (they lost most points for this, since changing the order of integration was the point of the exercise!)
Marking: 3 points for the sketch of the integration region, 3 points for writing the new integral correctly, 3 points for the integral w.r.t. x and 3 points for the final result.

5. The picture of the three surfaces involved is given below:



From the picture and the information given by the problem, it is clear that the variable z in the integration region takes values $-1/2 \leq z \leq 1/2$. Also, it is easy to see that $0 \leq \theta \leq 2\pi$, since the ellipsoid is centred at the z axis (at $z = 1$). The only variable for which the integration limits are not obvious is r. The values r can take must be determined by the ellipsoid's equation, which in cylindrical coordinates takes the form:

$$r^2 + \frac{(z-1)^2}{4} = 1 \quad \Rightarrow \quad r = \frac{1}{2}\sqrt{4-(z-1)^2}.$$

Therefore $0 \leq r \leq \frac{1}{2}\sqrt{4-(z-1)^2}$.

The integral that we have to compute is:

$$V = \int_{\theta=0}^{\theta=2\pi} d\theta \int_{z=-1/2}^{z=1/2} dz \int_{r=0}^{1/2\sqrt{4-(z-1)^2}} r dr,$$

where we included the Jacobian $J = r$ for cylindrical coordinates. The integral in r is:

$$\int_{r=0}^{1/2\sqrt{4-(z-1)^2}} r dr = [r^2/2]_{r=0}^{r=1/2\sqrt{4-(z-1)^2}} = \frac{4-(z-1)^2}{8}.$$

Plugging this into the z-integral we obtain

$$\frac{1}{8}\int_{z=-1/2}^{z=1/2}(4-(z-1)^2)dz = \frac{1}{8}[4z - \frac{(z-1)^3}{3}]_{z=-1/2}^{z=1/2} = \frac{1}{8}\left(2+\frac{1}{24}+2-\frac{27}{24}\right) = \frac{1}{2}-\frac{13}{96}.$$

Finally, the integral in θ is:

$$\int_{\theta=0}^{\theta=2\pi} d\theta = [\theta]_{\theta=0}^{\theta=2\pi} = 2\pi.$$

Therefore

$$V = 2\pi\left(\frac{1}{2}-\frac{13}{96}\right) = \pi\left(1-\frac{13}{48}\right) = 2.29074...$$

Feedback: Mostly ok, although some students had the integrals in the wrong order. ***Marking: 3 points for the r integration region. 4 points for writing down the correct integral and integrand. 3 points for the integral in z. 3 points for the integral in r and 1 point for the integral in θ. 3 points for the final result.***

(a) We try, as usual, solutions of the type $y = e^{mx}$. Putting this into the homogeneous equation we obtain

$$m^2 - 9 = 0 \quad \Leftrightarrow \quad m = \pm 3.$$

Therefore $u_1(x) = e^{3x}$ and $u_2(x) = e^{-3x}$. The Wronskian is

$$W(x) = u_1(x)u_2'(x) - u_1'(x)u_2(x) = -3e^{3x}e^{-3x} - 3e^{-3x}e^{3x} = -6.$$

(b) A particular solution of the inhomogeneous equation is given by

$$y_p = v_1(x)u_1(x) + v_2(x)u_2(x),$$

with

$$v_1(x) = -\int u_2(x)\frac{R(x)}{W(x)}dx, \qquad v_2(x) = -\int u_1(x)\frac{R(x)}{W(x)}dx.$$

Here $R(x) = \frac{6}{\sinh 3x/2}$ which gives

$$v_1(x) = \int \frac{e^{-3x}}{\sinh(3x/2)}dx = 2\int \frac{e^{-3x}}{e^{3x/2}-e^{-3x/2}}dx$$

changing variables to $t = e^{3x/2}$ (therefore $t^{-1} = e^{-3x/2}$) we have that $dt = (3/2)e^{3x/2}dx = (3t/2)dx$. So $dx = 2dt/(3t)$. and

$$v_1(x) = \frac{4}{3}\int \frac{t^{-2}}{t(t-t^{-1})}dt = \frac{4}{3}\int \frac{1}{t^2(t^2-1)}dt = \frac{4}{3}\int \left(\frac{1}{2(t-1)} - \frac{1}{2(t+1)} - \frac{1}{t^2}\right)dt$$

$$= \frac{2}{3} \ln \left| \frac{t-1}{t+1} \right| + \frac{4}{3t} = \frac{2}{3} \ln \left| \frac{e^{3x/2}-1}{e^{3x/2}+1} \right| + \frac{4e^{-3x/2}}{3}.$$

$$v_2(x) = -\int \frac{e^{3x}}{\sinh(3x/2)} dx = -\frac{4}{3} \int \frac{t^2}{t(t-t^{-1})} dt = -\frac{4}{3} \int \frac{t^2}{(t^2-1)} dt = -\frac{4}{3} \int \left(1 + \frac{1}{2(t-1)} - \frac{1}{2(t+1)} \right) dt$$

$$= -\frac{2}{3} \ln \left| \frac{t-1}{t+1} \right| - \frac{4t}{3} = -\frac{2}{3} \ln \left| \frac{e^{3x/2}-1}{e^{3x/2}+1} \right| - \frac{4e^{3x/2}}{3}.$$

Therefore, the general solution of the inhomogeneous equation is given by

$$y = c_1 e^{3x} + c_2 e^{-3x} + \frac{2e^{3x}}{3} \ln \left| \frac{e^{3x/2}-1}{e^{3x/2}+1} \right| + \frac{4e^{3x/2}}{3} - \frac{2e^{-3x}}{3} \ln \left| \frac{e^{3x/2}-1}{e^{3x/2}+1} \right| - \frac{4e^{-3x/2}}{3}$$

$$= c_1 e^{3x} + c_2 e^{-3x} + \frac{4\sinh(3x)}{3} \ln \left| \frac{e^{3x/2}-1}{e^{3x/2}+1} \right| + \frac{8\sinh(3x/2)}{3}.$$

for c_1, c_2 arbitrary constants.

Feedback: This was generally ok, although many students struggled to do the integrals and there were many typos and basic mistakes there.
Marking: 2 points for u_1, u_2. 3 points for the Wronskian. 1 point for the right formulae for v_1, v_2, 5 points for each integral and 2 points for the general solution.