Computational Mathematics

Modules MA1611 & AS1054

Laboratory Worksheets

Dr G Bowtell
City University

email: g.bowtell@city.ac.uk
web: www.staff.city.ac.uk/g.bowtell

Contents

Preface

This book of worksheets will be used throughout the course and you will be required to bring it with you when you attend the Computer Laboratory sessions. The object of the laboratory sessions is to supplement the lectures, building on material already covered and in a few instances introducing new material.

There will be four worksheets associated with each term and associated with each of the first six worksheets there will be a short assessment sheet which is to be handed in. Although copies of the assessment sheets are enclosed in this booklet fresh copies will be handed out in the laboratory sessions.

The Mathematics students will have one laboratory period per week and each Actuarial Science student two periods every other week. **Laboratory sessions will not start until week three of each term.**

Although each session will be staffed by a tutor you may wish to carry out the exercises at home or at some other time. If you do this you must ensure that your assessment sheet reaches the tutor assigned to your allocated laboratory session by the Monday of the following week it was due. My hope is that this flexibility will assist your studies and not present you with an opportunity to let things slide.

All notes and worksheets are available in electronic format on my web page

Dr G Bowtell

1 Introduction to Derive

Worksheet 1

This sheet introduces the package Derive for windows. This package is a useful mathematical tool which we will use throughout the course. In this first sheet you will learn how to use Derive to investigate functions of one variable: plotting the graph of $f(x)$; finding roots of $f(x) = 0$; estimating where local maxima and minima occur; examining asymptotes; differentiating and integrating $f(x)$;

Task 1 Entering Derive

Logon to the system using your id and password.

From the *Start* menu pull down the *Program* menu and then the *D.Maths and Statistics* menu. Select the option *Derive 5* .
(*Please open Derive5 rather than any older version of Derive that may still be on the system*)

Task 2 Inputting and plotting a function $y = f(x)$.

Derive refers to inputting an expression as **Authoring**. This can be done directly by left clicking with the mouse into the long thin **Authoring Window** at the bottom of the screen, (*just above all the symbols etc*) or alternatively by selecting **A**uthor from the top pull down menu and selecting **E**xpression.

Enter $10x \wedge 2 + 6x - 7$ into the authoring window and strike the enter key.

**Note** the $\wedge$ symbol is obtained by pressing [Shift] 6.

You should now see something like

$\#1 : 10x^2 + 6x - 7$

This means that the package has entered the function as expression 1, denoted by $\#1$.

Common to most window packages, there is usually more than one way to perform any one action. From now on, these notes will generally only tell you one route to achieve a result, but be aware that the route given may not be the fastest and you should experiment with the different menus and shortcut buttons.

Now plot the function as follows:

- With the expression highlighted click on the plot 2-D button ⌢, you should see a new display with a set of axes and a new set of menus and short cut buttons.
- Click again on the 2-D plot button. (6th *button from the left*)

You should see the graph of the curve $10x^2 + 6x - 7$.

Task 3 Viewing a curve

Note that the graph produced in task 2 is displayed over a range which is preset by Derive. We may want to chose a different scale in the x or y-direction or both and may want to chose a

different position of the origin. The current scale and the position of the origin are listed at the foot of the screen.

Select **S**et, **P**lot Range from the top menu bar. Change the horizontal minimum range to -8 and the maximum to +8. Change the vertical minimum range to -20 and the maximum to +20.

Also try experimenting with the following **Zooming** shortcut buttons to see what effect they have on the scales.
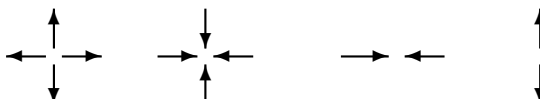
Task 4 Moving the cross

Move the mouse to position the cursor at some point on the screen and click. You should see the position of the cross move to the point you have clicked. Notice that the coordinates of the cross are given at the bottom of the screen. The cross can also be moved using the arrow keys on the keyboard.

What happens when you click on:

Task 5 Locating points of interest on the curve - **curve tracing**

Place the cross near the curve and then click on the trace button ⌒• (*9th button from the left.*)

The cross becomes a small box located on the curve. This can be moved along the curve using the keyboard arrows ← and →. The coordinates of the box are located in the bottom left hand corner of the plot.

Move the box cursor to the point on the curve where $x = 0$ and check that $y = -7$.

Now move the cursor to the local minimum to try and estimate its coordinates. By zooming in to improve the accuracy, estimate the coordinates of this point to one decimal place. Write your result below.

Coordinates of the minimum are (......................................, ...) to one decimal place.

Now look at the two points where the curve crosses the x-axis. (*You may need to reset the plot range or use the zoom buttons to recapture your picture.*) These points give the values of the roots of the quadratic, that is the solutions of $10x^2 + 6x - 7 = 0$. Move the tracing cursor to each point in turn and estimate the value of the roots to two decimal places. Write your answer below

The roots of $10x^2 + 6x - 7$ are $x_1 = $ ....and $x_2 = $ ....to two decimal places.

Check your answers using the formula for solutions of a quadratic equation, that is,

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Task 6 Using Derive to calculate the roots

Return to the first screen containing the algebraic expression $\#1{:}10x^2 + 6x - 7$ by clicking the rightmost button on the menu bar or by pulling down the **Window** menu and selecting the **Algebra1** option.

Pull down the **S**olve menu and select the **E**xpression option. (*Derive assumes you are solving the expression equal to zero*)

You should see a new options window.

- Select the method of solution to be algebraic (*this gives exact answers*) as opposed to the numerical option which give approximate answers.
- Select complex rather than real. This will solve the problem using complex numbers if necessary
- Click the **OK** button

You should now see the expression SOLVE $(10x^2 + 6x - 7, x)$

With this expression highlighted click **S**implify followed by **B**asic from the top menu bar. The two exact roots should now be displayed, note that they are connected by the logical or symbol

$$\bigvee$$

The expressions for the roots to $10x^2 + 6x - 7 = 0$ obtained by Derive are
$x_1 =$ ....and $x_2 =$ ....

Find a numerical value to three decimal places for x_1 and x_2 by highlighting the results and then selecting **Approximate** from the **Simplify** menu. A box should appear asking for the accuracy you wish Derive to use to carry out the evaluation (*leave this set at 10*). Click on **Approximate**.

The numerical values for the roots to $10x^2 + 6x - 7 = 0$ obtained by Derive are
$x_1 =$ ....and $x_2 =$ ....to three decimal places.

Compare the values that you obtained above with those you obtained from the estimates you made tracing the graph.

Now tidy everything up ready for a new task. Highlight expressions you wish to delete with the mouse button and select **D**elete from the **Edit** menu or use the shortcut button χ (*eighth button from the left*).

Task 7 Studying a more complicated curve

Enter the new expression $x \wedge 5 + 9x \wedge 4 + 10x \wedge 3 - 60x \wedge 2 - 56x + 96$ into the authoring window and strike the enter key to obtain:
$\#1 : x^5 + 9x^4 + 10x^3 - 60x^2 - 56x + 96$.

This is a polynomial of degree 5 (a quintic). Plot the curve as follows:

- Enter the plot window and using the delete shortcut button χ delete the graph of the quadratic function studied in tasks 2 to 6.

- Plot the graph using the plot button ⌁

- Set the plot range manually as before or just play around with the zoom buttons and the positioning buttons.(*ie origin at centre of screen button or cross at centre of screen button*)

Satisfy yourself that the curve has two local maxima, two local minima and that the function has five real roots (i.e. the curve crosses the x-axis five times). Check that the curve crosses the y-axis at 96.

Estimate the position of each turning point to one decimal place by tracing and zooming in for greater accuracy. List your answers below.

Coordinates of first maximum are (...., ...).

Coordinates of first minimum are (...., ...).

Coordinates of second maximum are (...., ...).

Coordinates of second minimum are (...., ...).

Similarly estimate where each root occurs and write down the values below.

$x_1 = $ ...., $x_2 = $ ...., $x_3 = $ ....,

$x_4 = $ ...., $x_5 = $ ....

Task 8 Simplifying the function

Return to the algebra window.

Make sure that the quintic expression is highlighted and select **Factor** from the **Simplify** menu. A box giving you five options, trivial, squarefree, rational, radical and complex, should appear. Select rational and then click on the **Factor** button.

A second expression should appear with the quintic written in factorised form, a simpler form than the original form. You should be able to read off the exact values for all the roots trivially. Check these values with those estimated from the graph of the function.

Task 9 Asymptotes of a function.

Again tidy everything up, removing any expressions and graphs you have generated. Enter the function.
$(x^3 + x^2 - 4x - 3)/(x^2 - 4)$

If you cannot remember how to do this, look back at tasks 2 and 7. This curve has two vertical asymptotes (are you able to see what they are from the form of the function?)

Plot out the function as before.

Experiment with the scale to get a good view of the curve. You should see that the curve is divided into three separate pieces. It crosses the x-axis at three distinct points and the y-axis at one point. Use the trace cursor to find the positions of the two vertical asymptotes.

Position of the first asymptotes is $x =$ ....

Position of the second asymptote is $x =$ ....

Check using the trace cursor that the curve crosses the y-axis at $y = 0.75$. Again using the trace cursor find the three positions where the curve crosses the x-axis. Hence list the three roots of $(x^3 + x^2 - 4x - 3)/(x^2 - 4) = 0$

First root is $x =$ ....

Second root is $x =$ ....

Third root is $x =$ ....

Return to the algebra window and use solve to find the roots of the function algebraically. Derive should give you three complicated looking expressions. Numerical approximations to these expressions can be obtained by highlighting and selecting **Approximate** from the **Simplify** menu. List the results below.

Derive calculates the roots to be $x =$ ...., $x =$ ....and $x =$ ....

Compare these answers with the ones you obtained graphically.

You should see that the curve has a local maximum and a local minimum. Find the coordinates of these points to one decimal place.

The position of the local maximum is (...., ...).

The position of the local minimum is (...., ...).

Looking at the curve you should notice that it approaches a straight line as $x \to \pm\infty$, that is, it has another asymptote in addition to the two vertical ones discussed above. Looking at the function for large positive and negative x, satisfy yourself that it approaches the line $y = x + 1$. To get a better idea of this, return to the algebra window and enter the new expression

$x + 1$

Then return to the plot screen and plot. You should now see both your original curve and the line $y = x + 1$. You should see that the curve really does approach $y = x + 1$ as $x \to \pm\infty$.

Task 10 (*omit if short of time*)

Investigate the behaviour of the function $(x^3 - 2x^2 + 2)/(x - 2)$ using Derive. Check out any local maxima and minima, real roots and asymptotes. How does the function behave as $x \to \pm\infty$.

Task 11 Differentiating a function.

- Enter the expression $x^3 + 5x^2 + 4x - 9$. If you cannot remember how to do this, then look back at Task 2.

- Select **D**ifferentiate from the **C**alculus menu. A window should appear.
  (*make sure that the variable of differentiation is set to x and the number of times the function is to be differentiated, ie the order, is set equal to 1*)

- Click on **S**implify or **OK** within this window. If you click **OK** Derive displays $\frac{d}{dx}$ of the function and you then have to select **B**asic from the **S**implify menu.

Check that your final answer is the derivative of expression 1.

Task 12 Solving for stationary points.

Stationary points occur when $f'(x) = 0$. Therefore, we can find the stationary points of $x^3 + 5x^2 + 4x - 9$ by highlighting the expression for its derivative and selecting **E**xpressions from the **S**olve menu. Select **A**lgebraically and **C**omplex and click the OK button. Use **S**implify **B**asic to produce the two exact roots. Approximations to these solutions can be found by highlighting the expression and selecting **A**pproximate from the **S**implify menu and then clicking on **A**pproximate within the window that appears.

Write down the values you obtain below.

The values of x at the stationary points are $x =$ ....and $x =$ ....
3 decimal places.

Task 13 Classifying the stationary points.

In order to classify whether the stationary points found in Task 3 are maxima or minima we need to find the second derivative of the original function and evaluate it at each stationary point in turn. If the second derivative is positive then a minimum occurs, whereas if the second derivative is negative a maximum occurs.

Highlight the original expression for the function. Select **D**ifferentiate from the **C**alculus menu. Change the **O**rder to order 2. Click on **S**implify. An expression for the second derivative should appear.

To evaluate the second derivative at each stationary point in turn:

- Highlight the expression for the second derivative.

- Select **V**ariable **S**ubstitution from the **S**implify menu. A window should appear.

The variable we are replacing is indicated as x, for functions of more than one variable we can use this option to select other variables

- Enter the value of x for the first turning point   *see note below.*

- Click on **S**implify and the value of the second derivative at the given x should appear.

- This may appear as a nasty fraction; to approximate this expression select **S**implify, **A**pproximate and then **A**pproximate again to give the value.

- a positive results indicates a local minima and a negative value a local maxima. Zero tells us nothing.

Note: Use the mouse buttons to highlight the required value of x, left click into the input box and then click the right mouse button and select the option to transfer the value - quicker than using the keyboard.

Repeat this exercise for the second turning point and hence find

$x_{MAX} =$ ....and $x_{MIN} =$ ....

Use the **V**ariable **S**ubstitution facility to evaluate $f(x)$ at each turning point.

Hence,

$y_{MAX} =$ ....and $y_{MIN} =$ ....

Task 14 Plot the curve

Plot a graph of the above function and check that your analysis is indeed correct. ie you have correctly located and classified the turning points.

Task 15 Integrating a function

- Highlight the original function $f(x) = x^3 + 5x^2 + 4x - 9$.
- Select **I**ntegrate from the **C**alculus menu.
- You are given the option of calculating either a definite or an indefinite integral. First select the **I**ndefinite option and click on **S**implify or **OK**.

*As with the derivative, Derive either calculates the answer straight away, if you use **S**implify, or displays $\int$ of the function if you click **OK**. In the latter case you then have to evaluate using **B**asic from the **S**implify menu.*

Check that the result given by Derive agrees with what you would expect. Notice that the options set the constant of integration equal to zero, strictly speaking you should always include one when you find an indefinite integral.

Now try evaluating a definite integral, for example $\int_{-5}^{6} x^3 + 5x^2 + 4x - 9 \ dx$. Again highlight the function $f(x) = x^3 + 5x^2 + 4x - 9$. Select **I**ntegrate from the **C**alculus menu. Select **D**efinite and insert the value 6 for the upper limit and -5 for the lower limit. Click on **S**implify.

An answer will appear, although you may need to use **A**pproximate from the **S**implify menu to obtain a decimal result. Write your answer below.

$$\int_{-5}^{6} x^3 + 5x^2 + 4x - 9 dx = \ ....$$

Check that this is the the expected result.

Task 16 A few additional exercises.

Find the turning points of the following functions using differentiation. Check your results are reasonable by plotting the curves.

1. $f(x) = x + \dfrac{1}{x}$

2. $f(x) = \dfrac{2}{x} + \ln x$

Evaluate the following integrals.

1. $\displaystyle\int_4^6 \dfrac{13x^2 - 8x + 5}{5x^2 - 16x + 3} dx \quad = \quad$ ....

2. $\displaystyle\int_6^3 \dfrac{(x-3)^3}{(x-2)^4} dx \quad = \quad$ ....

You might like to consider how you would evaluate these integrals analytically - more of this in the Maths course.

1.1   Assessment Exercises Worksheet 1

Surname:............................   **Forename**............................   **Group No**[1]. $\boxed{}$

For the Actuarial Science students this sheet is to be handed in at the associated lab session. For Maths students it should be handed in at the second of the two associated lab sessions. Late submission is only allowed with the permission of the lab tutor.

1. For task 7 write down the values of x where the curve cuts the x axis:

   $x_1 =$ ...., $x_2 =$ ....,

   $x_3 =$ ...., $x_4 =$ ....,

   $x_5 =$ ....

2. For task 12 write down the expression for the derivative of $f(x)$.

   $f'(x) =$ ....

   and correct to three decimal places the value of x at the stationary points;

   $x =$ ....and $x =$ ....

3. From task 15 complete, correct to three decimal places

   $$\int_{-5}^{6} x^3 + 5x^2 + 4x - 9 \, dx = ...$$

---

[1]Actuarial Science Students only

2   Iteration and Excel

Worksheet 2

This sheet uses Excel and the method of iteration to solve the problem $f(x) = 0$. It introduces user functions and self referencing to carry out the iterations.

Notation for instructions

[Enter] press the enter key

<C4> means make C4 the active cell

Task 1

We first look at the solution of the simple quadratic $f(x) = x^2 - 5x + 1 = 0$ using the simple rearrangement

$$x = g(x) = \frac{1 + x^2}{5}.$$

Show that this is indeed a rearrangement of $f(x) = 0$.

Task 2 User function

To implement an iterative process of the form $x_n = g(x_{n-1})$ we first create the function $g(x)$ within Excel. This is done using the Visual Basic Editor as follows:

Select **Tools** from the top menu.

Select **Macro**

Select **Visual Basic Editor**

A new display should appear, with a new top menu.

Select **Insert** from the top menu.

Select **Module**.

You should now have a blank window with 'Module 1' highlighted along the top. Now type the following piece of code:

function g(x) [Enter]

g=(1+x*x)/5 [Enter]

end function [Enter]   *note: this line appears automatically*

Now return to Excel 'Sheet 1' using the Excel button on the top menu.

In this sheet we implement two method for carrying out the iteration. The first is a direct method where you are able to view all the elements of the iterative sequence $\{x_n\}$, however you do have to carry out the iteration manually. The second method uses the ability for Excel to self reference cells and hence carry out the iteration automatically. This method has the advantage of being quick, also there is a facility to cause the iteration to stop automatically when a given accuracy has been achieved.

Task 3 Iteration Method 1: direct method

- To create a header line enter the following:

  <A1> x(n) [Enter]

  <B1> g(n) [Enter]

  The header is not essential, but it helps you to remember what each column refers to.

- To set up the iteration scheme enter the following.

  <A3> = 0.5 [Enter]   (starting guess x_0)

  <B3> = g(A3) [Enter]   (0.25 should appear: $g(x_0)$)

  <A4> = B3 [Enter]   (0.25 should appear: $x_1 = g(x_0)$)

  <B4> = g(A4) [Enter]   (0.2125 should appear: $g(x_1)$)

  The fixed point iteration scheme has now been set up.

- Executing the iteration to evaluate the roots.

  We can now perform as many iterations as we like as follows:

  - Highlight squares A4-B4.
  - Point the cursor to the bottom right hand corner of B4 until it changes to a +.
  - Drag the mouse down to A10:B10 and release.

  You should see the values of x_n in column A.

- Looking at the accuracy.

  We need to have some idea whether the sequence is converging. If it is converging, when do we decide to stop? To do this we look at the difference between successive values of x. Add a new column to your table as follows.

  <C1> error [Enter]

  <C4> =ABS(A4-A3) [Enter]   (0.25 should appear)

  To calculate the difference for all the iterates point the cursor to the bottom right hand corner of C4 until it changes to a +. Drag the mouse down to C10 and release. You should see that the numbers in column C are getting smaller and smaller. This suggests that the sequence x_0, x_1 ... is converging. The number in cell C10 gives an estimate for the error in the x value given in A10.

  Now try changing the initial value in <A3>. and see what happens. Can you make the scheme converge to a different root?

  What happens if you try a starting condition of 2.0? 3.0? 4.0? 5.0?

To obtain the second root to the quadratic it is necessary to use a different rearrangement of $f(x) = 0$. The next tasks asks you to do this, make the necessary changes to the user function g and then obtain the second root.

Task 4 Change the function $g(x)$ [2]

Convince yourself that a different rearrangement of $f(x) = x^2 - 5x + 1$ is

$$x = \frac{10x - 1}{x + 5}.$$

Change your function $g(x)$ to $(10x - 1)/(x + 5)$. Find the second root of $f(x) = 0$ correct to seven decimal places. (that is to say iterate until the error difference in column C is less than 0.5×10^{-7}.

Task 5 New problem.

Consider finding all the roots of the equation $f(x) = x - 2\sin^2(x) = 0$ using the method of rearrangement and iteration.
Show that each of the following are rearrangements of $f(x) = 0$:

- $x = \dfrac{x + 2\sin^2(x)}{2} = g(x)$

- $x = \sin^{-1}\sqrt{\dfrac{x}{2}} = g(x)$

Plot $y = f(x)$ using Derive and hence write down the solutions of $f(x) = 0$ correct to one decimal place. Use these values as your starting values and the rearrangements above to carry out the iterative process in Excel to obtain the two non-zero roots correct to five decimal places.

In your Excel Module for $g(x)$ the function $\sin^{-1}(x)$ is entered as **worksheetfunction.asin(x)** and $\sqrt{x}$ is entered as **sqr(x)**.

Having to find different rearrangements for different roots is very tedious. Newton's method for constructing the iterative process gets round this problem so long as your starting value x_0 is sufficiently close to the root you are trying to find. Apart from having to start near the required root the only other difficulty with Newton's method is that $f'(x)$ may become zero when carrying out the iteration or even worse it may equal zero at the root. In the following we will use the self referencing capability of Excel to carry out the iterations.

Task 6 Newton's formula

Show on paper that Newton's iterative formula for the solution of the equation

$$f(x) = 4\cos x - x = 0$$

is given by

$$x_n = \frac{4x_{n-1}\sin x_{n-1} + 4\cos x_{n-1}}{4\sin x_{n-1} + 1} = g(x_{n-1})$$

---

[2]instead of deleting the line $g = (1 + x \wedge 2)/5$ turn it into a comment line by inserting a $'$ symbol at the start of the line. Then write another line with g=(10*x-1)/(x+5). Doing this will keep a record of all the g's you have used.

Task 7 Enter the user function $g(x)$ into Excel

On you Excel spreadsheet change the user function $g(x)$ to

$$g(x) = (4*x*\sin(x)+4*\cos(x))/(4*\sin(x)+1).$$

Task 8 Iteration Method 2: self reference.

Excel allows us to **opt** for self reference. That is to say we can for example enter =1/(1+A1) into cell A1. This is a very direct form of self reference since we are trying to explicitly put an expression containing A1 in to cell A1. In this case Excel will carry out a default iteration for $x_n = 1/(x_{n-1} + 1)$ with $x_0 = 0$. Since this is somewhat restricting we employ a variation of this method that allows us to select x_0 before the iteration starts. Carry out the following to set up the above problem for iteration using $x_0 = 1$

- Clear the cells from row 3 down. ie just leave the headers

- To set the option for self reference select **Tools, Options, Calculation**. Tick the **iteration box**. Set the **max iterations** equal to 1 and set **maximum change** =0.001.

- In cell A3 enter 1, our starting value x_0 and in cell B3 enter =g(A3). In cell C3 enter =abs(A3-B3). At this stage A3 contains x_0 and B3 contains x_1, C3 contains $|x_0 - x_1|$ and there is no self reference. Finally in cell A3 type =B3.
  At this point in the process you have an indirect self reference and Excel will start iterating. Since we have set the maximum number of iterations equal to 1 it will only carry out one step. Keep pressing **F9** to repeat the iteration until you have obtained the desired accuracy. *Note that the convergence is very fast.*

- The iteration can be made more automatic if we increase the number of **maximum iterations**. The iteration will then continue until either it has carried out the maximum number of iteration or $|\frac{x_n - x_{n-1}}{x_{n-1}}| <$ maximum change.

- By plotting $y = f(x)$ using Derive, find two other starting values that give two other roots of the equation. Use your spreadsheet to find these roots.

Task 9 Final problem

Using Derive[3] to help with the derivative and the algebra show that the Newton iterative scheme for

$$f(x) = (x^2 - 1) - 5x \sin^2 x = 0$$

is given by:

$$x_n = \frac{10x_{n-1}^2 \sin x_{n-1} \cos x_{n-1} - x_{n-1}^2 - 1}{10x_{n-1} \sin x_{n-1} \cos x_{n-1} + 5\sin^2 x_{n-1} - 2x_{n-1}}$$

---

[3]if $f(x)$ is on line #10 and $f'(x)$ is on line #11 then you simply author **x - #10 / #11** to produce the Newton iteration formula. **Simplify, Basic** will then do the necessary algebra.

Plot $f(x)$ to obtain suitable starting values for x_0 in order to obtain all the roots of $f(x) = 0$.

Change your function $g(x)$ in Excel and hence using the above self referencing method obtain all the solution of $f(x) = 0$ correct to seven decimal places.

2.1   Assessment Exercises Worksheet 2

Surname:.............................   Forename..............................   **Group No[4].**

For the Actuarial Science students this sheet is to be handed in at the associated lab session. For Maths students it should be handed in at the second of the two associated lab sessions. Late submission is only allowed with the permission of the lab tutor.

1. For Task 1 write out the steps to show that $x = g(x) \quad \Rightarrow \quad f(x) = 0$.

2. For Task 5 draw a sketch for $y = f(x)$ indicating the non-zero roots of $f(x) = 0$ correct to 1 decimal place.

   The non-zero roots, x_1 and x_2 of $f(x) = 0$ correct to five decimal places are:

   $x_1 =$ .... $x_2 =$ ....

3. Write out your solution to Task 6

4. From Task 8 the three solutions to $f(x) = 4\cos x - x = 0$ correct to five decimal places are:

   $x_1 =$ .... $x_2 =$ .... $x_3 =$ ....

---

[4]Actuarial Science Students only

3   Solution of Linear and Non-Linear Equations - Excel
Worksheet 3

The first part of the sheet deals with the solution of linear simultaneous equations by first writing them as matrix equations and then using the matrix worksheet functions in Excel to obtain a solution. **The method is restricted to the case where we have the same number of equations as unknowns and the equations have a unique solution.** In the second part of the sheet Newton's method for the solution of N non-linear equations in N unknowns is investigated. You will need to use Derive and Excel for these exercises.

Notation for instructions

[**Enter**] press the enter key

[**Ctrl-Shft-Ent**] holding down simultaneously the Control key and the Shift key press the enter key. This is used to input an array variable or array function in Exel having first highlighted the recipient range.

<C4> means make C4 the active cell

Part 1

Task 1 Matrix operations in Excel

Given the following matrices:

$$A = \begin{pmatrix} 1 & 0 & 4 \\ -2 & 4 & 1 \\ 0 & 4 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & -1 & 7 \end{pmatrix}, \quad C = \begin{pmatrix} 3 & 4 \\ -2 & 1 \\ 3 & 8 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 1 \\ -2 & 3 \\ 3 & -7 \end{pmatrix}.$$

enter into the spreadsheet as follows:

<A1> A=[Enter],   enter the elements of A into cells B1:D3

<F1> B=[Enter],   enter the elements of B into cells G1:I1

<K1> C=[Enter],   enter the elements of C into cells L1:M3

<O1> D=[Enter],   enter the elements of D into cells P1:Q3

Carry out the following and write down the results below:

- <A5> inverse=[Enter],   Highlight B5:D7
  type =**MINVERSE(B1:D3)** [**Ctrl-Shft-Ent**]   *This gives A^{-1}*

- <A9> I =[Enter],   Highlight B9:D11
  type =**MMult(B1:D3,B5:D7)** [**Ctrl-Shft-Ent**]   *This gives AA^{-1}*

- The transpose of a matrix interchanges rows and columns, thus the result of the transpose of a 3×2 matrix will be a 2×3 matrix. This process is often needed when dealing with matrices, Excel therefore provides the following function.

To calculate the transpose of C, denoted C^T:

<K5> transpose=[Enter], Highlight L5:N6
type **=Transpose(L1:M3)** **[Ctrl-Shft-Ent]**   *This gives C^T*

- Although it is not possible to calculate AB it is possible to carry out the product BA. This follows from the fact that the number of columns of B is the same as the number of rows of A. The result will be a 1×3 matrix. See footnote[5] for general case. Calculate the following:

  <A13> BA=[Enter], Highlight B13:D13
  type **=MMult(G1:I1,B1:D3)** **[Ctrl-Shft-Ent]**   *This gives BA*

Task 2

Excel allows us to manipulate matrices using the usual arithmetic signs **+, −, ***. However *** is NOT** matrix multiplication, it merely multiplies together corresponding terms in each of the arrays. For this to make sense you must ensure that A and B have an equal number of rows and an equal number of columns. Note that this is different from normal matrix multiplication [5].

To calculate $C + D$, $2 * C$ and $C * D$.

- <K8> C+D=[Enter], Highlight L8:M10
  type = L1:M3+P1:Q3 **[Ctrl-Shft-Ent]**   This gives $C + D$

- <O8> 2C=[Enter], Highlight P8:Q10
  type = 2*(L1:M3) **[Ctrl-Shft-Ent]**   This gives $2C$

- <K12> C*D=[Enter], Highlight L12:M14
  type = (L1:M3)*(P1:Q3) **[Ctrl-Shft-Ent]**   This gives $C * D$

  warning this is not matrix multiplication, indeed it is not possible to multiply C and D using the normal matrix product, since the number of columns of one does not equal the number of rows of the other.

Task 3 Solution of equations using Matrix inverse

Start a new worksheet.

Write the following equations in the matrix form $A\underline{x} = \underline{b}$

$$
\begin{aligned}
x + 2y + z &= 1 \\
x - 2y + z &= 5 \\
3x - y + 2z &= 8
\end{aligned}
$$

Enter the matrix A and the column $\underline{b}$ into the new worksheet as follows:

---

[5]For normal matrix multiplication if A is an $m \times k$ matrix then B must be a $k \times n$ matrix in order to form AB. That is to say the number of columns of A must equal the number of rows of B. The result AB it an $m \times n$ matrix.

- \<A1\> A= [Enter],   enter the elements of A into cells B1:D3

- \<F1\> b= [Enter],   enter the elements of $\underline{b}$ into cells G1:G3

- \<I1\> inverse= [Enter],   Highlight J1:L3
  type = MINVERSE(B1:D3) [**Ctrl-Shft-Ent**]   *This gives* A^{-1}

- \<N1\> sol = [Enter],   Highlight O1:O3
  type = MMULT(J1:L3,G1:G3) [**Ctrl-Shft-Ent**]   *This gives* $A^{-1}\underline{b}$ *which equals* $\underline{x}$

  Hence write down the solutions $x =$ $y =$ $z =$

Task 4

Use the method of Task 3 to solve the following:

(a)
$$\begin{aligned} x + 2y - z + w &= 1 \\ x - y - 2z - w &= -8 \\ 2x - y + 3z + w &= 13 \\ -4x - 2y + 3z + w &= 3 \end{aligned}$$

(b)
$$\begin{aligned} x - y - z &= 1 \\ -x + y + 2z &= 0 \\ 2x - y + z &= 6 \end{aligned}$$

(c)
$$\begin{aligned} x + 2y + z + u + v + w &= 11 \\ x - y - z + 2u + v &= 3 \\ 3x - 2y + u &= 1 \\ x + 2z + 3w &= 9 \\ z + u &= 3 \\ 2y + u + w &= 8 \end{aligned}$$

Write down the solutions:

(a) $x =$ $y =$ $z =$ $w =$

(b) $x =$ $y =$ $z =$

(c) $x =$ $y =$ $z =$ $u =$ $v =$ $w =$

Part 2

In this section we use Excel to numerically find the solution of nonlinear simultaneous equation. The self reference property and the iteration that follows can be applied to matrices, thus Newton's extended method is appropriate.

Task 5

Solve $z = f(x,y) = 2 - 4x^2 - 2xy^2 = 0$ and $z = g(x,y) = 4xy - y^2 - 1 = 0$

Before trying to solve these equations we need to know approximately where the solutions lie. Derive is capable of plotting curves, not only when y is expressed explicitly in terms of x as $y = f(x)$, but also when y is expressed implicitly in terms of x as in this problem.

Use Derive to plot the curves given by $2 - 4x^2 - 2xy^2 = 0$ and $4xy - y^2 - 1 = 0$ as follows:

(You will need to enter $= 0$ when inputting the expressions)

- Author the equations $2 - 4x^2 - 2xy^2 = 0$ and $4xy - y^2 - 1 = 0$

- Switch to the plot window and plot in the usual way.
- Write down approximations to the root (*1 dec place is sufficient*) There should be three roots in total between $x = -1$ and $x = 1$

Hence: $\text{root}_1 = ($,   ) $\text{root}_2 = ($,   ) $\text{root}_3 = ($,   )

The first step of the scheme is given by

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} - \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}_{(x_0,y_0)}^{-1} \begin{pmatrix} f \\ g \end{pmatrix}_{(x_0,y_0)}$$

thus we have to first obtain the partial derivatives of the two functions. This can be done by hand or you can use Derive.

Partial derivatives using Derive

This is carried out in the same way as for ordinary differentiation with the only difference that if the function has more than one variable Derive will give you the option of choosing which variable you wish to differentiate with respect to. For the above functions:

- author the function $2 - 4x^2 - 2xy^2$
- Select **C**alculus **D**ifferentiate
- Choose the variable to be x, select **S**implify
  You will now have $f_x(x, y)$
- repeat the process but choose the variable to be y to give $f_y(x, y)$.
- author the function $4xy - y^2 - 1$ and repeat the above steps to obtain $g_x(x, y)$ and $g_y(x, y)$.

Write down the partial derivatives:

$f_x(x, y) =$ $f_y(x, y) =$ $g_x(x, y) =$ $g_y(x, y) =$

Hence complete:

$$J = \begin{pmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}$$

Iteration using Excel

Start a new worksheet

The following guides you through the implementation of the iterative scheme with the starting values $x_0 = -0.7$ and $y_0 = -0.5$.

- Construct two *user defined functions* in Excel. To do this open a VBA module, as in the last worksheet. Enter the following functions for $f(x, y)$ and $g(x, y)$.

function f(x,y)	function g(x,y)
f=2-4*x∧2-2*x*y∧2	g= 4*x*y-y∧2-1
End function	End function

- On the new worksheet enter the following
  <A1>   Xo= [Enter]   <B1>   =-0.7 [Enter]
  <A2>   Yo= [Enter]   <B2>   =-0.5 [Enter]
  <A3>   Matrix J [Enter]
  <A4>   * [Enter]   <B4>   * [Enter]
  <A5>   * [Enter]   <B5>   * [Enter]
  where * indicates the expressions for the four partial derivatives found above. Remember that on the sheet x is replaced by **B1** and y by **B2**.

  <A6>   Inverse J [Enter]
  Highlight cells **A7:B8**
  type **=MINVERSE(A4:B5) [Ctrl-Shft-Ent]**

  <A10>   f= [Enter]   <B10>   =f(B1,B2) [Enter]
  <A11>   g= [Enter]   <B11>   =g(B1,B2) [Enter]

  <A13>   X1=[Enter]   <A14>   Y1=[Enter]
  Highlight cells **B13:B14**
  type **=B1:B2-MMULT(A7:B8,B10:B11) [Ctrl-Shft-Ent]**

  At this point there is no self (circular) reference. To allow Excel to carry out self (circular) reference set the option as before, that is to say:
  - Select **Tools   Option   Calculation**
  - Tick **iterate**   set **maximum iterations** equal to 1 and **maximum change** equal to 0.001
  - Additionally when doing the iteration with matrices, as here, you should gain complete control of when the calculations are carried out. Tick the **manual** option in the *Calculation section* section.

- To start the iteration you create a self reference by entering the following:
  <B1>   =B13 [Enter]   <B2>   =B14 [Enter]
  Since we have selected manual no calculation will take place until you press the **F9** key. One step of the iteration will now take place. To carry out the iterations repeatedly press the **F9** key until the input values of x and y (cells B1 and B2) agree, to the desired accuracy, with the output values of x and y (cells B13 and B14).

  Solution: $x =$................ $y =$..............

To find the other two solutions enter the starting values in **B1** and **B2** followed by the **F9** key. Remember that after changing these cells to the new starting values you have to change them back to **=B13** and **=B14** and repeatedly press the **F9** key to carry out the iteration.

Task 6

Solve the following problems as in Task 5, first plotting the functions using Derive to locate and identify approximations to the solutions.

Warning: Since you have set the calculation option to manual your spreadsheet will only update its entries on pressing the **F9** key.

(a) $4x^2 - y^2(2x + y) = 0$ $y^4 - 4x^3 = 0$ (in addition to (0,0) this has two solutions)

(b) $x\cos y - y\cos x = 0$ and $1 - x^2 - y^2 = 0$ (this has two solutions)

(c) $xy - y^2 - z^2 = 0$ $x\sin(4\pi y) + 8zx - 1 = 0$ $32xyz - 1 = 0$

In this problem it is not possible to plot the curves as there are too many variables. Use the method extended to three variables to find the solution near to $x_0 = 0.2$, $y_0 = 0.2$ and $z_0 = 0.2$. Can you locate any other solutions by changing the starting values?

In the worksheet 4π is entered as 4*pi()
and in the VBA module as 4*WorksheetFunction.pi()

3.1   Assessment Exercises Worksheet 3

Surname:..........................   **Forename**...........................   **Group No**[6]. $\boxed{}$
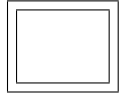
For the Actuarial Science students this sheet is to be handed in at the associated lab session. For Maths students it should be handed in at the second of the two associated lab sessions. Late submission is only allowed with the permission of the lab tutor.

1. Using the results of task 1 complete the following:

$$A^{-1} = \begin{pmatrix} & \\ & \end{pmatrix} \qquad AA^{-1} = \begin{pmatrix} & \\ & \end{pmatrix}$$

$$C^T = \begin{pmatrix} & \\ & \end{pmatrix} \qquad BA = (\qquad)$$

2. Using the results of Task 2 complete the following:

$$C+D = \begin{pmatrix} & \\ & \end{pmatrix} \qquad 2*C = \begin{pmatrix} & \\ & \end{pmatrix} \qquad C*D = \begin{pmatrix} & \\ & \end{pmatrix}$$

3. The solutions to the equations in Task 3 are

$x =$ .... $y =$ .... $z =$ ....

4. The solutions to the equations in Task 4(c) are:

$x =$ .... $y =$ .... $z =$ .... $u =$ ....

$v =$ .... $w =$ ....

5. For Task 5 complete:

$$J = \begin{pmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}$$

For Task 5 the three sets of solutions are given by:

(i) $x =$ .... $y =$ ....   (ii) $x =$ .... $y =$ ....

(iii) $x =$ .... $y =$ ....

---
[6]Actuarial Science Students only

4 Financial Functions in Excel

4.1 Supplementary Notes for Worksheet 4

The notation used in describing the various functions is as follows:

Notation

$$
\begin{aligned}
n &= \quad \text{total number of periods;} \quad \text{worksheet function NPER} \\
r &= \quad \text{interest rate per period;} \quad \text{worksheet function RATE} \\
\text{PMT} &= \quad \text{value of each payment made during the transaction} \\
\text{PV} &= \quad \text{present value} \\
\text{FV} &= \quad \text{future value} \\
\text{type} &= \quad \text{0 if the payments are at the end of each period.} \\
\text{guess} & \quad \text{starting value for iterative RATE function, set } =0 \\
i &= \quad \text{index of a period within the transaction} \\
\text{start} & \quad \text{period number at the start of a time period within the transaction} \\
\text{end} & \quad \text{period number at the finish of a time period within the transaction}
\end{aligned}
$$

Functions 1

The following functions are based on the single formula derived from the problem of making an initial deposit and then making further deposits at the end of each of a given number of periods. In terms of Excel's sign convention this is given by:

$$
\text{FV} = -\text{PV}(1+r)^n - \text{PMT}\left\{\frac{(1+r)^n - 1}{r}\right\} \tag{1}
$$

Excel has five separate functions based on this formula corresponding to making each of the variables in turn the subject of the formula. It is possible to do this explicitly for all but the rate r. For r Excel uses an iterative process with initial "*guess*" set by the user or if omitted set equal to zero by Excel; this default value of zero is usually successful.

=**FV**(r, n, PMT, PV, type)
=**PV**(r, n, PMT, FV, type)

=**PMT**(r, n, PV, FV, type)
=**NPER**(r, PMT, PV, FV, type)

=**RATE**(n, PMT, PV, FV, type, guess)   uses iterative process with first value=guess

Functions 2

Each payment, PMT, in a loan can be thought of as consisting of two parts; one part paying the current amount of interest charged and the remainder paying off some of the initial loan or principal. Excel has two functions that calculate these values for the ith payment:

=**IPMT**(r, i, n, PV, FV, type)   interest during the ith period
=**PPMT**(r, i, n, PV, FV, type)   principal paid off during the ith period

These are based on the formula (*see Ch 3*):

$$\text{IPMT} = -\left\{(1+r)^{i-1}\{\text{PV}r + \text{PMT}\} - \text{PMT}\right\} \quad \text{and} \quad \text{PPMT} = \text{PMT} - \text{IPMT}$$

The functions first calculate the PMT value before using these formula to calculate the actual values of interest paid and principal paid off in a given payment. Note that there is an option of setting FV to some value other than zero, thus these functions can be used in a loan problem where the original loan is not completely paid off.

Functions 3

Additionally Excel has two other functions that sum IPMT and PPMT for $i = $ start to $i = $ end. However these functions do assume that FV=0, ie the loan is always paid off by the end of the n periods.

=CUMIPMT(r, n, PV, start, end, type)   interest from "start" to "end" period inclusive
=CUMPRINC(r, n, PV, start, end, type)   principal paid off from "start" to "end" inclusive

These two functions carry out the following calculations:

$$\textbf{CUMIPMT}(r, n, PV, start, end, type) = \sum_{i=start}^{i=end} \textbf{IPMT}(r, i, n, PV, 0, type)$$

$$\textbf{CUMPRINC}(r, n, PV, start, end, type) = \sum_{i=start}^{i=end} \textbf{PPMT}(r, i, n, PV, 0, type)$$

These are useful for calculating the total amount of interest paid during a loan or during part of a loan, say over a given tax year.

4.2 Worksheet 4

Worksheet 4

Task 1

Enable the Analysis ToolPak as follows:

- Select Tools from the top menu
- Select Add-Ins
- Tick Analysis ToolPak

This allows you to use the CUM functions

Task 2

I open a new savings account with £100 and then continue to invest £25 per month thereafter.

(a) If the monthly rate is 0.3% how much do I have in the account after five years?

(b) If the monthly rate had been 0.5% how long, to the nearest month, would it take for the account to accrue at least £1000?

(c) If I wish to accrue £1500 in 4 years what interest rate should I be looking for?

(d) If the rate and term of my investment are as in (a) by how much should I increase my monthly payments to accrue £2500?

(e) Again if the rate and term are as in (a) and I still wish to save £25 per month, by how much should I increase my initial investment of £100 to accrue £2000 by the end of the five years?

Task 3

Sunil Spender a City University student decides to take out a loan to finance his latest purchase of a 4th generation mobile phone with built in widget connector. The loan is for £500 and he decides that he can afford to pay back £20 per month for as long as it takes. If the monthly rate is 1.75% how long does it take for him to make the repayment?

Task 4

If I borrow £1200 and pay back £40 per month for 3 years and my brother borrows £2400 and pays back £80 per month for 3 years, in terms of interest rates who has the better deal? Looking at Excel's formula for its financial functions is this the result you expected?

If in the above if my brother's loan had been over fours years, by how much worse, in terms of the difference in monthly interest rates, would his loan had been.

Task 5

On the first of February I take out a loan for £50000 at 0.4% per month making payments of £1000 per month at the end of each month until the loan is repaid.

- How long will it take to pay off the loan. The function NPER will not return an integer value thus we can do one of two things at the end. We can either make a smaller final payment at the end of the nth period or make a larger final payment at the end of the previous period. Consider each end case scenario giving in both cases the amount of the final payment.

- In this problem since equal payments do not render the loan to zero it is not possible to calculate the cumulative interest over a given period of the loan using the CUM functions, however we can us a series of IPMT functions with different i values and then sum the result. If the end scenario is that of making a larger earlier payment calculate the total amount of interest paid during the life of the loan as follows:

  - Set up a column of numbers from 1 to n, where n is the total number of payments made, starting at A3; label the head of this column in cell A1 as i
    (*you may of course do this anywhere on the worksheet*)
  - Use the extra amount paid in the final payment as the FV value (*make sure its entered as a negative*).
  - In B3 enter the function **=IPMT**(0.4%, A3, *your n*, 5000, *your* FV, 0). Copy down to include all i values. Sum this column to calculate the total interest.
  - To check that this calculation is indeed using the correct payments of £1000, in C3 enter the corresponding function PPMT for the principal paid and copy down to include all i. In each case the sum of the interest paid and the principal paid off should sum to £1000. Also the sum of the column of principals should give $-\{£5000+FV\}$.

- If the tax year is assumed to run from April to March inclusive calculate the interest paid and the amount of loan paid off in the first complete tax year of the loan.

  (*You may wish to enter a column of months alongside the interest payments, column* E *say, to identify the tax years; do this using the auto-fill method starting with* February *in* E3 *and* March *in* E4, *highlight both cells and use the + handle to copy down.*)

- Calculate the interest paid in the final tax year of the loan still assuming the larger payment final scenario.

Task 6

My Barclaycard, which I haven't used for a couple of months, is almost maxed out and I decide to cut it up and pay off my debts. I have £2500 owing and I decide to pay it off as quickly as I can, however I am not able to pay more than £100 per month. The monthly interest charge on the card is 1.75%. Calculate how many equal payments I should make on each due date to pay off the loan, and exactly how much these payments should be in order to meet my budget requirements. Calculate the amount of interest I have paid off at the end of the first month and how much interest I pay in the final payment.

A friend asks, how much did the loan cost me; calculate the total amount of interest charged over the complete life of the loan using the CUMIPMT function.

In this problem it is more accurate to set the "type" parameter =1

Task 7

When items are bought and they depreciate because of wear and tear and general inflation then we can use the above formulae to calculate the reduced final value by using a negative interest rate.

If I buy a car for £5000 (equivalent to investing £5000) and it depreciates at the rate of 0.5% per week how much will the car be worth in 2 years time (you may assume there are 104 weeks in two years)

4.3   Assessment Exercises Worksheet 4

Surname:...........................   **Forename**............................   **Group No**[7]. ▢
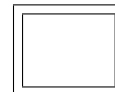
For the Actuarial Science students this sheet is to be handed in at the associated lab session. For Maths students it should be handed in at the second of the two associated lab sessions. Late submission is only allowed with the permission of the lab tutor.

1. For Task2(a) an acceptable way of writing out the solution is:

   The final value is given by: = **FV(0.3%,60,-25,-100,0)=£1760.48**

   Complete the following for (b) - (e) showing the financial function used along with its parameters.

   (b)   = ...   = ...

   (c)   = ...   = ...

   (d)   = ...   = ...   **increase =**   ....

   (e)   = ...   = ...   **increase =**   ....

2. For Task 4, and again writing out the financial functions used, complete the following:

   The interest on my loan   = ...   = ...

   The interest on my brother's loan   = ...   = ...

   The interest on my brother's loan over the longer period

   = ...   = ...

3. For Task 6:

   The number of payments   = ...   = ...
   (*this will not be a whole number of payments*)

   Hence the whole number of payments   = ...
   Using this value for the whole number of payments calculate the actual amount of each payment:

   = ...   = ...
   In this type of problem the payments are at the start of each period thus the "type" parameter should be set equal to 1

---

[7]Actuarial Science Students only

5   Polynomial Fitting and Linear Splines

Worksheet 5

Part 1

Polynomial Fitting

Problem 1

Given the following data use **Excel and the method below** to construct a unique polynomial $p(x)$, of degree at most five, through the six points. Hence plot the data and the polynomial on the same graph.

$x =$	0	0.1	0.2	0.4	0.5	0.7
$y =$	0	0.0841	0.1819	-0.3027	-0.4795	0.4599

Use your polynomial to estimate y at $x = 0.25$ and $x = 1$

Use Derive to compute the integral of $p(x)$ over the data range from 0 to 0.7.

Procedure for above

- Set up the following column headers so that you can keep track of the contents of your spreadsheet. In row1 columns A to H enter the symbols: x, y, a0, a1, a2, a3, a4, a5.

- Enter the x values in cells A3 to A8 and the y values in cells B3 to B8.

- In cells C3 to H8 we now form the matrix of coefficients of the a values. From the notes the coefficient of a_0 is always 1, the coefficient of a_1 is x, of a_2 it is x^2 etc.
  Enter:

=1 in cell C3 and copy down [8] to C8	*coefficients of a_0*
=A3 in cell D3 and copy down to D8	*coefficients of a_1*
=A3∧2 in cell E3 and copy down to E8	*coefficients of a_2*
=A3∧3 in cell F3 and copy down to F8	*coefficients of a_3*
=A3∧4 in cell G3 and copy down to G8	*coefficients of a_4*
=A3∧5 in cell H3 and copy down to H8	*coefficients of a_5*

- Before evaluating the a's we will label the cells next to where they will appear. In cells A10 to A15 enter a0=, a1=,... a5=.

- To calculate the a values in cells B10 to B15 we need to calculate the product of the inverse of the matrix in cells C3:H8 and the column of y data values in cells B3:B8. (*see lecture notes.*) We do this using the array functions MMULT and MINVERSE as follows:

  Highlight B10 to B15
  Enter =MMULT(MINVERSE(C3:H8),B3:B8)   **Ctrl-Shift-Enter**.
  B10 to B15 should now contain the values of a_0 to a_5 respectively.

- Write out the polynomial

  $p(x) =$ ....

---

[8] use the mouse to grab the + in the righthand bottom corner of C3 and drag down to C8

- To plot the data and the polynomial we first need to create a list of (x, y) values for $p(x)$.
  - In cells A18 to A53 enter the x values from 0 to 0.7 in steps of 0.02. You can use the auto fill method by first entering 0 in A18 and 0.02 in cell A19. Highlight A18:A19 and then copy down by grabbing the + handle in the righthand bottom corner of A19 and drag and drop to A53.
  - To enter $p(x)$ in cells B18 to B53. In cell B18 enter the polynomial as:
    =\$B\$10+\$B\$11*A18+\$B\$12*A18∧2+\$B\$13*A18∧3 +\$B\$14*A18∧4+\$B\$15*A18∧5
  - Copy down B18 to B53.
- To plot the two sets of data on the same chart first plot the six given data points just as points:
  - Highlight the data A3:B8
  - Select the chart wizard from the top menu bar.
  - Select XY (Scatter)
  - Select Chart sub-type for just plotting points; the first in the list. Select Finish This should give a chart with the six data points.
- To add the data for $p(x)$:
  - Highlight A18:B53 and select copy from the top Edit menu. (*or use the right mouse button or use Ctrl-C*)
  - Activate the outside boundary of the chart by left clicking just inside the chart.
  - From the top menu select Edit and Paste Special. In the dialogue box select: New Series; Columns; Categories (X Values) in First Column; OK.
  - Since the Chart type is just points the data will be plotted as points. To change this so that the points are not shown but joined by straight lines you need to:
    * Point to any data point on the chart, you should get a message when you do this, ensure that it says that you are pointing to series 2.
    * Right click and select Chart Type from the pop up menu.
    * Change the Chart sub-type to the one that joins points with straight lines but does not show the points, the last in the list. Select OK.

You should now have a chart showing the original six data points and a curve representing the polynomial that passes through the points. This polynomial can then be used to estimate the vales of y between the given data points, **interpolation**, or values of y outside the range of the data points, **extrapolation**.

Calculate:

$p(0.25) =$ .... $p(1.0) =$ ....

To calculate $\int_0^{0.7} p(x)\, dx$, **A**uthor $a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$ into **Derive** using the numerical values of the a's obtained above. Use Calculus Integrate with the definite integral option and the limits set equal to 0 and 0.7. Express your answer correct to three decimal places:

$$\int_0^{0.7} p(x)\, dx = ...$$

Problem 1a(*Comparison with exact function*)

The above data is in fact generated by $f(x) = x \sin(10x)$. Use Derive to plot this and the polynomial $p(x)$. You can now see how good an approximation $p(x)$ is to $f(x)$ inside the data range. Outside the data range the approximation is poor and gets worse as x moves further out. You should note that the two curves, as expected, cross at the data points.

Problem 2 (*for the speedy*) The following problem illustrates how using a polynomial may not give a good result. Consider the data:

$x =$	-1	0	1	2	3	4
$y =$	-0.455	-0.833	-5.000	1.250	0.556	0.357

- Construct the polynomial of degree of at most five through the above points.
- As in Problem 1 use Excel to plot the data points and the polynomial on the same chart.
- Use Derive to plot the polynomial.
- The above data is in fact generated from $f(x) = \dfrac{1}{x - 1.2}$. Plot this in Derive on the same graph as $p(x)$. We can now observe the difference between $f(x)$ and $p(x)$ in the data range. They agree at the data points but have very little else in common!

Part 2

Linear Spline fitting

Problem 3 (*Construct a Spline*)

Given the data points $\{(x_k, y_k) : k = 0, \ldots n\}$, with $x_0 < x_1 < \ldots < x_n$ and $y_k = f(x_k)$, we wish to approximate $f(x)$ by joining consecutive data points with linear polynomials (straight lines) i.e. linear splines.

The complete spline on $[x_0, x_n]$ is denoted by $S(x)$. On each sub-interval we denote $S(x)$ by $s_k(x), x \in [x_k, x_{k+1}], k = 0, 1, \ldots n - 1$.

The formula for the linear spline can be written down explicitly as

$$s_k(x) = a_k + b_k(x - x_k) \quad k = 0, 1, \ldots n - 1,$$

where in terms of x_k, x_{k+1}, y_k and y_{k+1} the coefficients

$a_k =$ ....   and $b_k =$ ....

Given the following table of values,

x_k	0	1	2	3	4
$y_k = f(x_k)$	6.00000	4.62091	1.75156	0.03002	1.03907

and using either a calculator or Derive calculate the following:

$a_0 =$ .... $b_0 =$ ....

$a_1 =$ .... $b_1 =$ ....

$a_2 =$ .... $b_2 =$ ....

$a_3 =$ .... $b_3 =$ ....

Hence write down,

$$S(x) = \begin{cases} s_0(x) = & \text{.....................} \quad \text{.....................} \quad x \in [\quad , \quad] \\ s_1(x) = & \text{.....................} \quad \text{.....................} \quad x \in [\quad , \quad] \\ s_2(x) = & \text{.....................} \quad \text{.....................} \quad x \in [\quad , \quad] \\ s_3(x) = & \text{.....................} \quad \text{.....................} \quad x \in [\quad , \quad] \end{cases}$$

In order to enter $S(x)$ as a single function into Excel so that it can be plotted we need to use "a compounded worksheet if statement" or a VBA programme containing the "if..then...else" structure. The VBA approach is clearer and therefore preferable, Problem 4 constructs such a VBA function and asks you to plot both S(x) and the given data points.

Problem 4 (*plotting and evaluating a spline*) Since by choosing the correct chart type Excel is already capable of plotting points and then joining them with straight line segments, this problem to this extent is redundant. However it does show how to construct a user function that is defined in a piece-wise fashion over several intervals and how to use it to produce values and draw a graph.

- Start a new VBA module and complete the following to produce the spline function $S(x)$ above:

  Function S(x as double)
  if x>=0 AND x<=1 then
  S=(*your expression for $s_0(x)$ obtained above*)
  elseif x>=1 AND x<=2 then
  $\vdots$
  else
  S="out of range"
  endif
  End Function

- Start a fresh worksheet and enter the five pairs of data values in cells A1:B5.
- Plot a new chart for these data values using the chart wizard with the XY (Scatter) and points only subtype.
- To construct the spline for the data enter values of x in cells A8:A28 from $x = 0$ to $x = 4$ in steps of 0.2 (*use the auto fill method as in Problem 1*) In cell B8 type =S(A8) and then copy down to cell B28.
- Add this set of data for $S(x)$ to your chart. (*Use the same method as in problem 1, remember to make sure that you also change the chart subtype for this set of data from just points, to join with straight lines.*)

Problem 5 (*Comparison with the actual function.*)

One of the applications of using splines or polynomials is to replace a difficult known function with a simpler one. The data given in Problem 3 actually came from the function $f(x) = 3(\cos x + 1)$.

To add the graph of $f(x)$ to our chart in Problem 4:

- In cell C8 enter =3*(cos(A8)+1) and copy down to cell C28.
- Highlight the values of x in cells A8:A28, hold down the control key and highlight the $f(x)$ values in cells C8:C28. Copy using the Edit menu or use the right-mouse button.
- The data is now saved and ready to be pasted onto the chart. Again use the method of Problem 1 with Paste Special selected from the Edit menu.
- Change the chart type for this data from points only to join by lines using the method in Problem 1.

The graph indicates how good an approximation the spline is to $f(x)$. We may be interested in the error between the spline and $f(x)$ at given values of x.

Calculate:

$|f(0.5) - S(0.5)| =$ ...., $|f(1.5) - S(1.5)| =$ ....

Problem 6 (*optional for the really dedicated*)

Use Derive or otherwise to calculate:

$$I_0 = \int_0^4 S(x)\ dx \quad \text{and} \quad I_1 = \int_0^4 f(x)\ dx$$

Hence complete:

$I_0 =$ .... $I_1 =$ .... $|I_0 - I_1| =$ ....

(*In Excel use the worksheet function* =**ABS** *to calculate* $|\ |$)

5.1   Assessment Exercises Worksheet 5

Surname:............................   **Forename**.............................   **Group No^9.** [ ]

This sheet is to be handed in at the lab session(s) associated with Worksheet 5

1. For Problem 1 on Worksheet 5 write down the following:

   (a) $p(x) =$ ....

   (b) $\int_0^{0.7} p(x)\, dx =$ ....

   (c) $p(0.25) =$ .... $p(1.0) =$ ....

2. For Problem 3 on Worksheet 5 write out the complete linear spline $S(x)$ with its intervals of definition:

$$S(x) = \begin{cases} S_0(x) = & \text{.......................................} & x \in [\quad,\quad] \\ S_1(x) = & \text{.......................................} & x \in [\quad,\quad] \\ S_2(x) = & \text{.......................................} & x \in [\quad,\quad] \\ S_3(x) = & \text{.......................................} & x \in [\quad,\quad] \end{cases}$$

3. For Problem 5 on Worksheet 5:

   (a) $|f(0.5) - S(0.5)| =$ ....

   (b) $|f(1.5) - S(1.5)| =$ ....

---
^{9}Actuarial Science Students only

6   Least Squares Fitting

Worksheet 6

Single variable linear fitting

Problem 1 (*construct fit*)

Find the least squares linear fit $y = m_1 x + c$, to the given data points as follows:

$x =$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$y =$	1	0.997	0.987	0.970	0.946	0.915	0.877	0.831	0.777	0.714	0.642

- Enter the x values in cells A1:A11 and the y values in cells B1:B11.
- Highlight cells A13:B13
- enter **=LINEST(B1:B11,A1:A11)** followed by **Ctrl-Shift-Enter**

Cell A13 should now contain m_1 and cell B13 contain c.

The linear fit is given by: $y =$ ....

Problem 2 (*plot data and fit*)

- Set up a column of y values using $y = m_1 x + c$:
  - Enter =$A$13*A1+$B$13 in cell C1
  - Copy this down to cell C11.
  - Cells C1 to C11 should now contain the predicted y values.
- Plot the original data:
  - Highlight cells A1 to B11.
  - Select the Chart Wizard.
  - Select Scatter
  - Select the sub type that produces just points (first in the list of subtypes)
  - Select Finish
- Plot the straight line fit on the same graph:
  - Highlight and Copy the y values C1 to C11
  - Click on the chart and Past the values just copied. The points will be plotted with the same x values as the last set of data. (*this is the simplest way of adding data to a chart and is appropriate when the new data has exactly the same set of x values as the existing set.*)
  - To change the points to a straight line display **right-click** on one of the new data points (*make sure this is flagged as Series 2*);
    Select Chart Type; Select the subtype that give only straight lines. (*the last of the options.*) Select OK.

Does this look like a good approximation?   ....

Polynomial Fitting

Problem 3 (*Add polynomial Trendline*)

- To add a polynomial fit to the data on the chart Excel provides us with the Add Trendline option.
    - Activate the chart by left clicking just inside the boundary.
    - Select Add Trendline from the Chart Menu on the top menu bar
    - Select Polynomial and Change the order to 3. OK

The "best fit" cubic should now be shown on your chart. At this stage the equation of the cubic is not known thus proceed as follows:

Problem 4 (*Cubic fit*)

To fit the polynomial $y = p_3(x) = m_3x^3 + m_2x^2 + m_1x + c$ to the data in problem 1:

- Construct three columns, one for each of x, x^2 and x^3 and one column for the y data:
    - Copy the x data from A1:A11 to A17:A27
    - Enter =A17∧2 in cell B17. Copy down to B27
    - Enter =A17∧3 in cell C17. Copy down to C27
    - Copy the y data from B1:B11 to D17:D27
- Find the parameters using LINEST:
    - Highlight cells A30:D30
    - enter **=LINEST(D17:D27,A17:C27)** followed by **Ctrl-Shift-Enter**

Cell A30 should now contain m_3, cell B30 m_2, cell C30 m_1 and cell D30 contain c.

The cubic polynomial fit is given by: $y = p_3(x) =$ ....

Problem 5 (*Plot data and cubic*)

As follows, first plot the data on a new chart; create a set of data points derived from the cubic; add the new data to the chart.

- Create new chart with the original data
    - Highlight the original data in A1:B11 and plot a new chart as in Problem 2 selecting points only from the chart subtype.
- Create new values using the cubic.
    - Since we need to plot the cubic at more than the given eleven points, in order to see its true shape, we need to create two new columns of x-y values. Do this at x intervals of 0.02 as follows:
    - In cells F17:F67 enter the x values 0 to 1.0 in steps of 0.02 using the auto fill method. (*see the procedure to Problem 1 on Worksheet 5*)
    - In cell G17 enter the polynomial:
      = $D$30+$C$30*F17+$B$30*F17∧2+$A$30*F17∧3
    - Copy cell G17 down to G67 to give the column of y values generated from the cubic.
- Add new data to chart.

– Having created a new set of points at different x values to the original data set adding these points to the chart is slightly more involved than in Problem 2. (*this is covered in the procedure to Problem 1 on Worksheet 5, or as follows*)

– Highlight the points to be added; F17:G67. Select Copy from the top Edit menu or use the right mouse button and the Copy option or use *Ctrl-C*.

– Activate the Chart by left clicking just inside the boundary of the chart.

– From the Edit menu select Paste Special. In the dialogue box select: New Series; Columns; Categories (X Values) in First Column: OK

– Change the Chart type for this set of points to join with lines without points as in Problem 2 above.

Does this look like a good approximation?   ....

Problem 6 (*coefficient of determination*)

LINEST also provides us with statistics that enable us to judge how good an approximation we are making. The easiest indicator to consider is the coefficient of determination, denoted r^2, where r is the correlation coefficient. Recall the closer r^2 is to $+1$ the better the fit. To make LINEST display the statistics we need to use:
=LINEST(_known y_**,** _known x_**, TRUE, TRUE)**

- To find r^2 for Problem 1. (*linear fit*)
    – LINEST requires five rows and two columns to output all its results.
    Highlight E1:F5.
    Enter **=LINEST(B1:B11,A1:A11, TRUE, TRUE)** followed by:
    Ctrl-Shift-Enter
    – The first row contains m_1 and c as before. The second row contains the standard errors. The third row contains r^2 and sey.
    $r^2 = $ ....
- To find r^2 for Problem 4. (*cubic fit*)
    – Since we are determining four parameters, m_1, m_2, m_3 and c, LINEST will require five rows and four columns to output all its results.
    Highlight H1:K5
    – Recall that the *y-values* are in D17:D27 and the *x-values* are in A17:C27, thus:
    Enter **=LINEST(D17:D27,A17:C27, TRUE, TRUE)** followed by:
    Ctrl-Shift-Enter
    – As before r^2 is in the third row, first column:
    $r^2 = $ ....

Are the above values of r^2 in agreement with your comments regarding the goodness of fit based on the two graphs above?   ....

Problem 7 (*an application*)

The above data is in fact generated from the function $f(x) = \dfrac{x}{\tan x}$ with $f(0) = 1$ and rounded to three decimal places. We can use the polynomial in many application, for example to approximate the function at an intermediate point or as here, to evaluate an integral.

Evaluate
$$\int_0^1 p_3(x)\ dx \quad \text{to approximate} \quad \int_0^1 \frac{x}{\tan x}\ dx$$

by entering $p_3(x)$ into Derive and carrying out the definite integral.

$$\int_0^1 p_3(x)\ dx = \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

Enter the function $\dfrac{x}{\tan x}$ into Derive and hence carry out an approximate integration.

$$\int_0^1 f(x)(x)\ dx \approx \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

Multi-variable fitting

Problem 8 (*simple problem*)

In a sense polynomial fitting is an example of multi-variable fitting where the extra variables are generated from the single x variable using various powers of x. This problem is a straightforward multi-variable fitting problem where there is no given relationship between the sets of x-data.

Enter the data

$x_1 =$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$x_2 =$	-1	3.0	4.0	2.0	5.0	3.0	1.0	2.0	-4.0	5.0	1
$x_3 =$	1.0	3.0	5.0	7.0	9.0	7.0	5.0	3.0	2.0	1.0	0.0
$y =$	0.01	6.20	9.40	9.43	14.81	10.84	6.97	6.23	-0.40	7.96	3.01

into the first four columns of a new worksheet.

Use LINEST with the options set to **TRUE** to find

The values of m_3, m_2, m_1 and c such that $y = m_3 x_3 + m_2 x_2 + m_1 x_1 + c$ is the least squares fit to the data.

The statistical data for the goodness of fit.

By considering the value of r^2 do you consider this to be a good fit?   ....

Use the fit to approximate y at $x_1 = 0.35$, $x_2 = 4.5$ and $x_3 = 6.0$. $y = $ ....

Problem 9 (*linear combination of functions*)

Adapt the method of fitting a polynomial to fit the function

$$y = m_4 \cos(2x) + m_3 \sin(2x) + m_2 \cos(x) + m_1 \sin(x) + c \tag{2}$$

to the data:

$x =$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$y =$	0	0.366	0.895	1.640	2.671	4.077	5.976	8.516	11.887	16.334	22.167

That is to say construct four columns of x-values generated from $\sin x$, $\cos x$, $\sin 2x$ and $\cos 2x$ and one column for the y data. Apply LINEST to the y column and the four x-value columns.

From the value of r^2 comment on the goodness of fit.

$r^2 =$ .... and hence the fit is good/bad.

As in the above problems construct a Chart consisting of the original data plotted as points. Plot also the y values calculated from Eq(2) for $x = 0$ to 1 in steps of 0.02.

Does the graph seem to be in agreement with your comment above based on r^2?   ....

General Fitting (*using Solver*)

You will need to ensure that the Solver Add-In is active on your sheet. To check this pull down the Tools menu and see if Solver is listed. If not then from the Tools menu select Add-Ins and tick the Solver Add-In check box.

Problem 10

This question asks you to use Solver to find the values of a and b such that $y = axe^{bx}$ is a least squares fit to the data in Problem 9.

To use Solver we need to specify a target cell containing the sum of the squares and the cells that contain the two parameters a and b.

- Enter the x - *values* in A1:A11 and the y - *values* in B1:B11
- Enter 1 in F1 and 1 in G1   *initial values of a and b*
- Enter =($F$1*A1*EXP($G$1*A1)-B1)∧2 in C1 and copy down to C11
- Enter =SUM(C1:C11) in C13

The sheet is now ready to use solver to minimise the contents of cell C13 (*the sum of squares*) subject to variation of cells F1 and G1 (*the values of a and b.*)

- Select solver from the Tools menu.
- Make target cell equal to C13
- Select Equal to: Min
- Make sure that the By Changing Cells: box contains F1:G1 or indeed $F$1:$G$1, either will do.
- Solve.

Complete: $a =$ .... $b =$ ....   Sum Squares $=$ ....

As in Problem 9 plot a chart of the data and the fitted function $y = axe^{bx}$, in steps of 0.02, with the above values of a and b.

Does the graph indicate a good fit?   ....

6.1   Assessment Exercises Worksheet 6

Surname:.............................   **Forename.............................**   **Group No**[10]. $\boxed{}$

This sheet is to be handed in at the lab session(s) associated with Worksheet 6

Express answers correct to four decimal places

1. For Problem 1 write down the linear fit: $y =$ ....

2. From Problem 4 write down the cubic fit

   $$y = p_3(x) = ...$$

3. From Problem 6 write down the values of r^2 for:

   (a) the linear fit in Problem 1:   ....

   (b) the cubic fit in Problem 4:   ....

   Which of these, (a) or (b), gives the best fit to the given data: ....

4. From Problem 9 use the values of m_1, m_2, m_3, m_4 and c to complete the following fit:

   $$y = (.................)\cos 2x + (.................)\sin 2x + (.................)\cos x + (.................)\sin x + (.................)$$

---

[10]Actuarial Science Students only

7   Cubic Splines & Minitab

Worksheet 7

Cubic Splines.

The linear spline is a poor approximation to $f(x)$ away from the knots (*data points*). We now consider the construction of a cubic spline through the following data points.

x_k	0	1	2	3	4
$y_k = f(x_k)$	6.00000	4.62091	1.75156	0.03002	1.03907

For this set of data the kth component of the spline is given by:

$$s_k(x) = a_k + b_k(x - x_k) + c_k(x - x_k)^2 + d_k(x - x_k)^3 \quad \text{for } k = 0, 1, \ldots 3.$$

- Each component has to pass through the points at the ends of the interval on which it is defined. Therefore,

$$\begin{aligned}
s_0(x_0) &= f(x_0), \\
s_0(x_1) &= f(x_1), \\
s_1(x_1) &= f(x_1), \\
s_1(x_2) &= f(x_2), \\
s_2(x_2) &= f(x_2), \\
s_2(x_3) &= f(x_3) \\
s_3(x_3) &= f(x_3), \\
s_3(x_4) &= f(x_4).
\end{aligned}$$

- The derivative of each component has to match the derivative of the next component at their common knot. Therefore,

$$\begin{aligned}
s_0'(x_1) &= s_1'(x_1), \\
s_1'(x_2) &= s_2'(x_2), \\
s_2'(x_3) &= s_3'(x_3).
\end{aligned}$$

- The second derivative of each component has to match the second derivative of the next component at their common knot. Therefore,

$$\begin{aligned}
s_0''(x_1) &= s_1''(x_1), \\
s_1''(x_2) &= s_2''(x_2), \\
s_2''(x_3) &= s_3''(x_3).
\end{aligned}$$

You can see that we have 14 conditions here. However, we have 16 coefficient values (a_k, b_k, c_k and d_k for $k = 0, 1, 2$ and 3) so we need two more conditions.

- The **natural** cubic spline is defined by specifying two extra conditions, namely,

$$s_0(x_0)'' = 0$$
$$s_3(x_4)'' = 0$$

Finding the coefficients

The calculation of the coefficients using the above conditions can be formalized by generalizing the approach used in the note. You are not expected to be able to derive or know the general method however you may be able to spot from the following example how the method works when you have more data points.

- a_0, a_1, a_2, a_3 and c_0
  Using the above data and the conditions $s_0(x_0) = f(x_0)$, $s_1(x_1) = f(x_1)$, $s_2(x_2) = f(x_2)$ and $s_3(x_3) = f(x_3)$, write down the coefficients a_k.

  $a_0 =$ ...., $a_1 =$ ...., $a_2 =$ ....

  Using $s_0''(x_0) = 0$ show that $c_0 = 0$.

- c_1, c_2 and c_3
  It **can** be shown (*for the case of equally spaced x values, as in this example*) that the remaining conditions imply that the coefficients c_1, c_2 and c_3 satisfy the matrix equation

$$\begin{pmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \frac{3}{h} \begin{pmatrix} \delta_1 - \delta_0 \\ \delta_2 - \delta_1 \\ \delta_3 - \delta_2 \end{pmatrix} \tag{3}$$

  where, δ_k is given by:

$$\delta_k = \frac{f(x_{k+1}) - f(x_k)}{h}$$

  and h is the interval of x spacings, which in this example is given by $h = 1$.
  Thus

$$\delta_k = f(x_{k+1}) - f(x_k)$$

  By direct hand calculation complete the following:

$$\frac{3}{h} \begin{pmatrix} \delta_1 - \delta_0 \\ \delta_2 - \delta_1 \\ \delta_3 - \delta_2 \end{pmatrix} = \frac{3}{h} \begin{pmatrix} \text{..............................} \\ \text{..............................} \\ \text{..............................} \end{pmatrix} = \begin{pmatrix} \text{..............................} \\ \text{..............................} \\ \text{..............................} \end{pmatrix}$$

  You now have the righthand side of Eq(3), hence use Excel to solve this matrix equation for c_1, c_2 and c_3.

  $c_1 =$ ....   , $c_2 =$ ....   and $c_3 =$ ....

- d_0, d_1, d_2 and d_3

  The d's in terms of the c's can be shown to be:

  $$d_0 = \frac{c_1}{3} = \quad \text{...} \qquad d_1 = \frac{c_2 - c_1}{3} = \quad \text{...}$$
  $$d_2 = \frac{c_3 - c_2}{3} = \quad \text{...} \qquad d_3 = -\frac{c_3}{3} = \quad \text{...}$$

- b_0, b_1, b_2 and b_3

  The b's are given by the following (*in our example $h = 1$*)

  $$b_0 = \delta_0 - c_0 h - d_0 h^2 = \delta_0 - c_0 - d_0 = \quad \text{........................}$$
  $$b_1 = \delta_1 - c_1 h - d_1 h^2 = \delta_1 - c_1 - d_1 = \quad \text{........................}$$
  $$b_2 = \delta_2 - c_2 h - d_2 h^2 = \delta_2 - c_2 - d_2 = \quad \text{........................}$$
  $$b_3 = \delta_3 - c_3 h - d_3 h^2 = \delta_3 - c_3 - d_3 = \quad \text{........................}$$

You now have all the coefficients. Hence write down the cubic spline.

$$S(x) = \begin{cases} s_0(x) = \quad \text{...} \\ s_1(x) = \quad \text{...} \\ s_2(x) = \quad \text{...} \\ s_3(x) = \quad \text{...} \end{cases}$$

Plot the data points and the cubic spline

- Construct two columns containing just the data points.
- Plot a graph of the data points using the chart type that consists only of point (*first on the list*)
- Write a VBA function module for $S(x)$.
- Construct a column of x values on your worksheet in the range $x = 0$ to $x = 1$ in steps of 0.02. (Use auto fill facility to produce these values)
- Use your VBA function to construct a corresponding column of $S(x)$ values next to the column of x values.
- Copy and Paste these points to the graph using Paste Special.
- Change the data type for these points to 'join with lines only'. This is achieved by right clicking on a data point for this series and selecting chart type. The required chart type is the last in the list.

Minitab

Open files

Open Minitab which is under programs on the Act Sci Departmental Software.

Retrieve the Excel Workbook Lab7data on Dr Bowtell's staff web page

www.staff.city.ac.uk/g.bowtell $\rightarrow$ Computation IT (2008-2009)$\rightarrow$ Lab7data

Save this data file in your home directory - **do not try and work directly with it**.

Open the Excel workbook so retrieved.

The Excel workbook contains two sheets, sheet 2 containing the results from a first year progress test separated into two groups and sheet 1 containing the combined data in ascending order.

Copy data

To copy the marks from Excel sheet(1) to column 1 (C1) of Minitab:

- Highlight A1:A108 in sheet 1 of the Excel workbook and copy.
- In Minitab with row1-col1 active Paste the data using the right mouse option or Paste from the pull down Edit menu.
- In the "label" row immediately under C1 name the column   total.

Repeat the above process to copy the Math and ActSci data from sheet 2 of the Excel Workbook to respectively C2 and C3 of Minitab. Label the data as Math and ActSci

Descriptive Statistics

Minitab allows us to calculate a set of simple statistics for a given data set and displays the results on the Session window.

- Select **Stat** from the top menu and **Basic Statistics** from the sub menu followed by **Display Descriptive Statistics**.
- Set **Variable set** equal to C1 C2 C3 by typing these directly into the window, or use the names you have given the data or highlight each variable in turn and press the Select button.
- Select **Statistics** and tick: mean; SE of mean; standard deviation; variance; coefficient of variation; Min; Max; First quartile; Median; Third quartile; interquartile range; N non missing.
  Select OK OK to close all sub windows.
- The basic statistics should now be visible in the **Session Window**

Enter the results:

Data	N	Mean	SE	StDev	Var	CoefVar	Min	Q_1	Med	Q_3	Max	IQR
Total												
Math												
ActSci												

Table 1:

Box Plots

Simple box plots can be used to compare the three sets of data, indicating central tendency using the median and spread using quartiles, whiskers and outliers. To plot a box plot for each of the three sets of data on the same graphs with the boxes horizontal carry out the following:

- Select **Graph** from the top menu followed by **Box Plot**.
- Select **Multiple Y - Simple** OK
- Select all three data sets as the **Graph variables**.
- By default Minitab creates vertical box plots, here we produce horizontal box plots by:

Selecting **Scale** and ticking **transpose**. OK. OK

This should give a graph containing three box plots placed horizontally. Some of the data used to create the plots can be displayed by pointing into the relevant box or by simply reading from the x-axis.

With reference to task 3 check that the ends of the boxes are at Q_1 and Q_3 with the median in the correct place. Check with reference to the data that the position of any outliers and the extreme ends of the arms are shown correctly.

Frequency Histogram

Minitab allows three types of histogram, frequency, percentage and density. This task creates a simple frequency histogram for each of the three sets of data using the default 'equal intervals'. It then asks you to place these graphs together with the box plot in a single layout.

- Select **Graph** from the top menu followed by **Histogram**.
- Select Simple type. OK
- Set **Graph variables** to C1 C2 C3. Select **Scale**
- Select the **y-scale type** tag and select **Frequency**. OK. OK

This gives a default Histogram for each of the three data sets of the selected type with equal subintervals over a slightly extended range.

To produce a layout of the four graphs created so far:

- With one of the histograms active (*click inside one of the histograms windows if not active*) Select **Editor** from the top menu followed by **Layout tools**.
- The layout tool is fairly easy to follow. Select two rows and two columns and position your three Histograms and Box Plot. Select **Finish**

To print out or save the layout:

- Make the layout active
- Select **File** from the top menu
- Select **Print Graph** or **Save Graph As**

Percentage and Density Histograms

This tasks requires you to construct a percentage and a density histogram using unequal class intervals (sub intervals) for the data in C1.

Close all the existing graph windows using the $\times$ in the right hand corner of each window and select NO to save.

- Select **Graph**, **Histogram Simple**, OK
- Enter just C1 as the Graph variable
- Select **Scale**, **Y-Scale type**, **Percent** OK OK

Repeat the above but select **Density** as the Y-Scale type.

You should now have two histograms that are the same shape, only the Y-scales are different in value. This is to be expected as the subintervals on which the histograms are based are all of equal length. In such a case is it irrelevant as to whether it is the height or area of each rectangle that is proportional to the frequency.

You are now required to change the x-Scale of these graphs to create histograms with unequal class intervals. The two histograms should then appear different, as the percentage histogram is based on height whereas the density histogram is based on area.

With the percentage histogram active:

- Select **Editor** from the top menu, **Select Item**, **X-Scale**. (*an alternative to this is to point to the x-axis, until you get the message X-scale, and left click*)
- Select **Editor** from the top menu, Select **Edit** *x*-**Scale**. (*alternatively point to the selected x-axis and right-click; select option*)
- We wish to change the subintervals (*class intervals*) on which the histogram is based, this is referred to as "binning".
  Click the **binning** tag.
- We can specify the start and ends of the intervals on which the rectangles are constructed, **cut-points** or alternatively specify the **midpoints**.
  Select: **Cut-point**. Select: **Midpoint/Cutpoint positions**. It is now possible to enter the cut-points in the window.
- Enter the values: 0 30 50 60 70 80 85 90 95 100 leaving a space between each. OK

The histogram will now be converted into one with the given class intervals.

Repeat the above process on the density histogram.

It is now possible to see the difference in shape between the two histograms due to the subintervals not being equal.

Layout

Use **Layout tools** (*see above*) to form a layout of your two histograms, setting the Layout tool to have two rows and one column. Print or save your layout.

8   Markov Matrices and Revision Problems

Worksheet 8

This sheet contains three tasks on Markov matrices followed by some revision questions. The style of this sheet is similar to that of the Practical Examination in the summer which also only uses Derive and Excel.

Task 1 (*Markov Matrix*)

Show that the matrix M given by:

$$M = \begin{pmatrix} 0.5 & 0.2 & 0.3 \\ 0.2 & 0.2 & 0.6 \\ 0.3 & 0.6 & 0.1 \end{pmatrix}$$

is a Markov matrix.

(i) If $\underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ has column sum α, that is to say $a_1 + a_2 + a_3 = \alpha$, show that $M\underline{a}$ also has column sum α.

(ii) Show that $M \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

(iii) If $\underline{v} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ calculate $M\underline{v}$ and $M^2\underline{v}$.

What would you expect $M^n\underline{v}$ to tend to as n increases?

Task 2 (*Markov Matrix*)

Given

$$M = \begin{pmatrix} \frac{1}{4} & \frac{1}{10} \\ \frac{3}{4} & \frac{9}{10} \end{pmatrix}$$

(a) Use MMULT(M,M), where M denotes the range containing M, to calculate M^2. If M^2 is in cell range A use MMULT(A,A) to obtain M^4.

(b) Evaluate $M^4\underline{x}_0$ where $\underline{x}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(c) Using Excel and the iterative scheme $\underline{x}_{k+1} = M\underline{x}_k$:

(i) Calculate $M^4\underline{x}_0$ and compare your result with (b).
(ii) Calculate $M^n\underline{x}_0$ as n tend to infinity.

Recall you will need to set the Tool-Options-Calculation page to allow iteration in single steps and select manual rather than automatic - this means the sheet recalculates only on pressing F9 - don't forget to change this back to automatic when finished
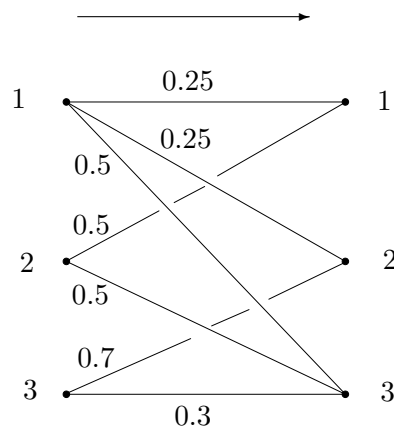
Figure 1:

Task 3 (*Markov Matrix*)

Write down the Markov matrix M given by fig 1

Hence use the method of Task 1 to find the limit of $M^n \begin{pmatrix} 0.5 \\ 0.25 \\ 0.25 \end{pmatrix}$ as n gets large.

Task 4 (*Iteration - single variable*)

- Show that
$$x = \sin^{-1}\left(\frac{(x+1)}{3}\cos x\right) = g(x)$$

  is a rearrangement of
$$f(x) = x + 1 - 3\tan x = 0$$

  .

- Use Derive to plot $y = x + 1$ and $y = 3\tan x$. Estimate the root of $f(x) = 0$ in the interval $0 < x < \frac{\pi}{2}$ correct to one decimal. Denote this root as α.
- Use Excel and the iterative scheme $x_{k+1} = g(x_k)$ starting with $x_0 = \alpha$ to obtain a root of $f(x) = 0$ correct to four decimal places.
- Write down Newton's iterative scheme for the solution of $f(x) = 0$. Hence again using $x_0 = \alpha$ find the solution of $f(x) = 0$ correct to thirteen decimal places.

Task 5 (*Iteration - two variables*)

Given:
$$f(x,y) = \cos x + \cos y - 1 \qquad g(x,y) = y - x^2$$

- Use Derive to plot $f(x,y) = 0$ and $g(x,y) = 0$. Hence obtain correct to one decimal place an approximation to the root in the region $0 \le x \le \frac{\pi}{2}$
- Write down the matrix:
$$J = \begin{pmatrix} f_x(x,y) & f_y(x,y) \\ g_x(x,y) & g_y(x,y) \end{pmatrix}$$

- Use Excel and Newton's iterative scheme in two variables to obtain the above root correct to five decimal places.
- From the symmetry of $f(x, y)$ and $g(x, y)$ and your plot try to write down some other roots to the equation.

Task 6 (*Simultaneous equations and matrix inverse*)

Given the equations:

$$
\begin{aligned}
2x + y + z + w &= 1 \\
x - y + 3z - w &= 1 \\
4x - 2y + z + w &= 3 \\
x - y - z - w &= 2
\end{aligned}
$$

Write in matrix form $A\underline{x} = \underline{b}$ and hence use Excel to to find A^{-1} and the solutions to the equations.

Task 7 (*Polynomial through points*)

Given the data:

$x =$	0.1	0.2	0.3	0.4	0.5
$y =$	0.96	0.83	0.62	0.36	0.07

write down five simultaneous equations in a_0, a_1, a_2, a_3 and a_4 such that the polynomial $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$ passes through the five points.

Use Excel to find $p(x)$ with the coefficients expressed correct to three decimal places.

Plot on the same chart the data as points and the polynomial evaluated from 0.1 to 0.4 in steps of 0.025 as a solid curve. (*see notes and previous sheets on how to add data to a chart*)

Task 8 (*Least squares - Linear multi variable*)

Given the following set of data use LINEST to find the best least squares fit for the function $y = c + m_1X_1 + m_2X_2 + m_3X_3$:

$y =$	0.8	2.01	2.46	5.49	4.34	13.25
$X_1 =$	0	0.1	0.2	0.3	0.4	0.5
$X_2 =$	1.0	1.5	2.5	3.0	4.5	6.0
$X_3 =$	−0.1	0.1	−0.2	0.3	−0.4	0.5

Write down the coefficient of determination (i.e r^2)

Task 9 (*Linear Least Squares fit using general functions*)

Given the data:

$x =$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$y =$	0	0.10	0.40	0.89	1.56	2.35	3.19	3.95	4.54	4.89	5.00

Use Excel and LINEST applied to suitable columns of data to find the best least squares fit of the function:

$$y = c + m_1 \cos x + m_2 \cos 2x + m_3 cos 3x$$

to the set of given data.

Write down the coefficient of determination and the y-standard error.

The above data is an approximation to

$$f(x) = \frac{10x^2}{1 + x^4}$$

use your least squares fit and Derive to calculate

$$\int_0^1 y \, dx \quad \text{as an approximation to} \quad \int_0^1 f(x) \, dx$$

Derive can carry out $\int_0^1 f(x) \, dx$ exactly so you can compare your approximation with the exact value.

task 10 (*Excel Financial Function*)

This subject will only occur in the practical test as part of a question where you are asked to do **either** *it* **or** *a question on a different subject.*

You will be required to use the Excel Financial Functions

If I borrow £2000 at 1.50% per month and make payments of £40 per month at the end of each month, how much will I owe after making 50 payments?

How long will it take me to pay off the loan if my final payment is not more than £40. What will the final payment be?

If I borrow £5000 over 40 months making payments of £150 per month at the end of each month, what is the monthly interest rate?