Computational Mathematics/Information Technology

Worksheet 2

Iteration and Excel

This sheet uses Excel and the method of iteration to solve the problem $f(x) = 0$. It introduces user functions and self referencing to carry out the iterations.

Notation for instructions

[Enter] press the enter key

<C4> means make C4 the active cell

Task 1

We first look at the solution of the simple quadratic $f(x) = x^2 - 5x + 1 = 0$ using the simple rearrangement

$$x = g(x) = \frac{1 + x^2}{5}.$$

Show that this is indeed a rearrangement of $f(x) = 0$.

Task 2 User function

To implement an iterative process of the form $x_n = g(x_{n-1})$ we first create the function $g(x)$ within Excel. This is done using the Visual Basic Editor as follows:

Select **Tools** from the top menu.

Select **Macro**

Select **Visual Basic Editor**

A new display should appear, with a new top menu.

Select **Insert** from the top menu.

Select **Module**.

You should now have a blank window with 'Module 1' highlighted along the top. Now type the following piece of code:

```
function g(x) [Enter]
g=(1+x*x)/5 [Enter]
end function [Enter]          note: this line appears automatically
```

Now return to Excel 'Sheet 1' using the Excel button on the top menu.

In this sheet we implement two method for carrying out the iteration. The first is a direct method where you are able to view all the elements of the iterative sequence $\{x_n\}$, however you do have to carry out the iteration manually. The second method uses the ability for Excel to self reference cells and hence carry out the iteration automatically. This method has the advantage of being quick, also there is a facility to cause the iteration to stop automatically when a given accuracy has been achieved.

Task 3 Iteration Method 1: direct method

- To create a header line enter the following:
  <A1> x(n) [Enter]
  <B1> g(n) [Enter]
  The header is not essential, but it helps you to remember what each column refers to.

- To set up the iteration scheme enter the following.
  <A3> = 0.5 [Enter] (starting guess x_0)
  <B3> = g(A3) [Enter] (0.25 should appear: $g(x_0)$)
  <A4> = B3 [Enter] (0.25 should appear: $x_1 = g(x_0)$)
  <B4> = g(A4) [Enter] (0.2125 should appear: $g(x_1)$)
  The fixed point iteration scheme has now been set up.
- Executing the iteration to evaluate the roots.
  We can now perform as many iterations as we like as follows:
  - Highlight squares A4-B4.
  - Point the cursor to the bottom right hand corner of B4 until it changes to a +.
  - Drag the mouse down to A10:B10 and release.

  You should see the values of x_n in column A.

- Looking at the accuracy.
  We need to have some idea whether the sequence is converging. If it is converging, when do we decide to stop? To do this we look at the difference between successive values of x. Add a new column to your table as follows.
  <C1> error [Enter]
  <C4> =ABS(A4-A3) [Enter] (0.25 should appear)

  To calculate the difference for all the iterates point the cursor to the bottom right hand corner of C4 until it changes to a +. Drag the mouse down to C10 and release. You should see that the numbers in column C are getting smaller and smaller. This suggests that the sequence x_0, x_1 ... is converging. The number in cell C10 gives an estimate for the error in the x value given in A10.
  Now try changing the initial value in <A3>. and see what happens. Can you make the scheme converge to a different root?
  What happens if you try a starting condition of 2.0? 3.0? 4.0? 5.0?

To obtain the second root to the quadratic it is necessary to use a different rearrangement of $f(x) = 0$. The next tasks asks you to do this, make the necessary changes to the user function g and then obtain the second root.

Task 4 Change the function $g(x)$ [1]

Convince yourself that a different rearrangement of $f(x) = x^2 - 5x + 1$ is

$$x = \frac{10x - 1}{x + 5}.$$

Change your function $g(x)$ to $(10x - 1)/(x + 5)$. Find the second root of $f(x) = 0$ correct to seven decimal places. (that is to say iterate until the error difference in column C is less than 0.5×10^{-7}.

[1] instead of deleting the line $g = (1 + x \wedge 2)/5$ turn it into a comment line by inserting a $'$ symbol at the start of the line. Then write another line with g=(10*x-1)/(x+5). Doing this will keep a record of all the g's you have used.

Task 5 New problem.

Consider finding all the roots of the equation $f(x) = x - 2\sin^2(x) = 0$ using the method of rearrangement and iteration.
Show that each of the following are rearrangements of $f(x) = 0$:

- $$x = \frac{x + 2\sin^2(x)}{2} = g(x)$$
- $$x = \sin^{-1}\sqrt{\frac{x}{2}} = g(x)$$

Plot $y = f(x)$ using Derive and hence write down the solutions of $f(x) = 0$ correct to one decimal place. Use these values as your starting values and the rearrangements above to carry out the iterative process in Excel to obtain the two non-zero roots correct to five decimal places.

In your Excel Module for $g(x)$ the function $\sin^{-1}(x)$ is entered as **worksheetfunction.asin(x)** and $\sqrt{x}$ is entered as **sqr(x)**. Hint: always type these function names in lower case. If you type them correctly the Excel will capitalise various letters automatically. If it doesn't then you have made a spelling error — a good check.

Having to find different rearrangements for different roots is very tedious. Newton's method for constructing the iterative process gets round this problem so long as your starting value x_0 is sufficiently close to the root you are trying to find. Apart from having to start near the required root the only other difficulty with Newton's method is that $f'(x)$ may become zero when carrying out the iteration or even worse it may equal zero at the root. In the following we will use the self referencing capability of Excel to carry out the iterations.

Task 6 Newton's formula

Show on paper that Newton's iterative formula for the solution of the equation

$$f(x) = 4\cos x - x = 0$$

is given by

$$x_n = \frac{4x_{n-1}\sin x_{n-1} + 4\cos x_{n-1}}{4\sin x_{n-1} + 1} = g(x_{n-1})$$

Task 7 Enter the user function $g(x)$ into Excel

On you Excel spreadsheet change the user function $g(x)$ to

$g(x)$ =(4*x*sin(x)+4*cos(x))/(4*sin(x)+1).

Task 8 Iteration Method 2: self reference.

Excel allows us to **opt** for self reference. That is to say we can for example enter =1/(1+A1) into cell A1. This is a very direct form of self reference since we are trying to explicitly put an expression containing A1 in to cell A1. In this case Excel will carry out a default iteration for $x_n = 1/(x_{n-1} + 1)$ with $x_0 = 0$. Since this is somewhat restricting we employ a variation of this method that allows us to select x_0 before the iteration starts. Carry out the following to set up the above problem for iteration using $x_0 = 1$

- Clear the cells from row 3 down. I.e., just leave the headers
- To set the option for self reference select **Tools, Options, Calculation**. Tick the **iteration box**. Set the **max iterations** equal to 1 and set **maximum change** =0.001.

- In cell A3 enter 1, our starting value x_0 and in cell B3 enter =g(A3). In cell C3 enter =abs(A3-B3). At this stage A3 contains x_0 and B3 contains x_1, C3 contains $|x_0 - x_1|$ and there is no self reference. Finally in cell A3 type =B3.
  At this point in the process you have an indirect self reference and Excel will start iterating. Since we have set the maximum number of iterations equal to 1 it will only carry out one step. Keep pressing **F9** to repeat the iteration until you have obtained the desired accuracy. *Note that the convergence is very fast.*
- The iteration can be made more automatic if we increase the number of **maximum iterations**. The iteration will then continue until either it has carried out the maximum number of iteration or $|\frac{x_n - x_{n-1}}{x_{n-1}}| <$ maximum change.
- By plotting $y = f(x)$ using Derive, find two other starting values that give two other roots of the equation. Use your spreadsheet to find these roots.

Task 9 Almost the final problem

Using Derive[2] to help with the derivative and the algebra show that the Newton iterative scheme for

$$f(x) = (x^2 - 1) - 5x\sin^2 x = 0$$

is given by:

$$x_n = \frac{10x_{n-1}^2 \sin x_{n-1} \cos x_{n-1} - x_{n-1}^2 - 1}{10x_{n-1} \sin x_{n-1} \cos x_{n-1} + 5\sin^2 x_{n-1} - 2x_{n-1}}$$

Plot $f(x)$ to obtain suitable starting values for x_0 in order to obtain all the roots of $f(x) = 0$.

Change your function $g(x)$ in Excel and hence using the above self referencing method obtain all the solution of $f(x) = 0$ correct to seven decimal places.

Task 10 The final bit

In your Excel Module define for $g(x)$ the function $\log x$. In your Excel 'Sheet 1' enter

<D3> = log(5) [Enter] (This should give you $\log 5$)

<D4> = g(5) [Enter] (This should give you $\log 5$)

[2] if $f(x)$ is on line #10 and $f'(x)$ is on line #11 then you simply author **x - #10 / #11** to produce the Newton iteration formula. **Simplify, Basic** will then do the necessary algebra.

Assessment Exercises Worksheet 2

Surname:.............................. **Forename..............................** **Group No[3].**

For the Actuarial Science students this sheet is to be handed in at the associated lab session. For Maths students it should be handed in at the second of the two associated lab sessions. Late submission is only allowed with the permission of the lab tutor.

1. For Task 1 write out the steps to show that $x = g(x) \quad \Rightarrow \quad f(x) = 0$.

2. For Task 5 draw a sketch for $y = f(x)$ indicating the non-zero roots of $f(x) = 0$ correct to 1 decimal place.

   The non-zero roots, x_1 and x_2 of $f(x) = 0$ correct to five decimal places are:

   $x_1 =$ .... $x_2 =$ ....

3. Write out your solution to Task 6

4. From Task 8 the three solutions to $f(x) = 4\cos x - x = 0$ correct to five decimal places are:

   $x_1 =$ .... $x_2 =$ .... $x_3 =$ ....

5. From Task 10 write out the two answers to 4 decimal places:

   $\log 5 =$ .... $\log 5 =$ ....

   Give a reason why this *is* sensible. (You won't get any marks for this reason, but if you come up with a really good answer you may get a job in Microsoft's PR department.)

[3] Actuarial Science Students only