

Computational Mathematics/Information Technology

Worksheet 3

The first part of the sheet deals with the solution of linear simultaneous equations by first writing them as matrix equations and then using the matrix worksheet functions in Excel to obtain a solution. **The method is restricted to the case where we have the same number of equations as unknowns and the equations have a unique solution.** In the second part of the sheet Newton's method for the solution of N non-linear equations in N unknowns is investigated. You will need to use Derive and Excel for these exercises.

Notation for instructions

[Enter] press the enter key

[Ctrl-Shift-Ent] holding down simultaneously the Control key and the Shift key press the enter key. This is used to input an array variable or array function in Excel having first highlighted the recipient range.

<C4> means make C4 the active cell

Part 1

Task 1 Matrix operations in Excel

Given the following matrices:

$$A = \begin{pmatrix} 1 & 0 & 4 \\ -2 & 4 & 1 \\ 0 & 4 & 5 \end{pmatrix}, \quad B = (2 \quad -1 \quad 7), \quad C = \begin{pmatrix} 3 & 5 \\ -2 & 1 \\ 3 & 8 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 1 \\ -2 & 3 \\ 3 & -7 \end{pmatrix}.$$

enter into the spreadsheet as follows:

<A1> A=[Enter], enter the elements of A into cells B1:D3

<F1> B=[Enter], enter the elements of B into cells G1:I1

<K1> C=[Enter], enter the elements of C into cells L1:M3

<O1> D=[Enter], enter the elements of D into cells P1:Q3

Carry out the following and write down the results below:

- <A5> inverse=[Enter], Highlight B5:D7
type =MINVERSE(B1:D3) [Ctrl-Shift-Ent] *This gives A^{-1}*
- <A9> I =[Enter], Highlight B9:D11
type =MMult(B1:D3,B5:D7) [Ctrl-Shift-Ent] *This gives AA^{-1}*
- The transpose of a matrix interchanges rows and columns, thus the result of the transpose of a 3×2 matrix will be a 2×3 matrix. This process is often needed when dealing with matrices, Excel therefore provides the following function.

To calculate the transpose of C , denoted C^T :

<K5> transpose=[Enter], Highlight L5:N6
 type =**Transpose(L1:M3)** [Ctrl-Shft-Ent] *This gives C^T*

- Although it is not possible to calculate AB it is possible to carry out the product BA . This follows from the fact that the number of columns of B is the same as the number of rows of A . The result will be a 1×3 matrix. See footnote¹ for general case. Calculate the following:

<A13> BA=[Enter], Highlight B13:D13
 type =**MMult(G1:I1,B1:D3)** [Ctrl-Shft-Ent] *This gives BA*

Task 2

Excel allows us to manipulate matrices using the usual arithmetic signs $+$, $-$, $*$. However ***** is **NOT** matrix multiplication, it merely multiplies together corresponding terms in each of the arrays. For this to make sense you must ensure that A and B have an equal number of rows and an equal number of columns. Note that this is different from normal matrix multiplication¹.

To calculate $C + D$, $2 * C$ and $C * D$.

- <K8> C+D=[Enter], Highlight L8:M10
 type = L1:M3+P1:Q3 [Ctrl-Shft-Ent] This gives $C + D$
- <O8> 2C=[Enter], Highlight P8:Q10
 type = 2*(L1:M3) [Ctrl-Shft-Ent] This gives $2C$
- <K12> C*D=[Enter], Highlight L12:M14
 type = (L1:M3)*(P1:Q3) [Ctrl-Shft-Ent] This gives $C * D$

warning this is not matrix multiplication, indeed it is not possible to multiply C and D using the normal matrix product, since the number of columns of one does not equal the number of rows of the other.

Task 3 Solution of equations using Matrix inverse

Start a new worksheet.

Write the following equations in the matrix form $A\underline{x} = \underline{b}$

$$\begin{aligned} x + 2y + z &= 1 \\ x - 2y + z &= 5 \\ 3x - y + 2z &= 9 \end{aligned}$$

Enter the matrix A and the column \underline{b} into the new worksheet as follows:

- <A1> A= [Enter], enter the elements of A into cells B1:D3

¹For normal matrix multiplication if A is an $m \times k$ matrix then B must be a $k \times n$ matrix in order to form AB . That is to say the number of columns of A must equal the number of rows of B . The result AB is an $m \times n$ matrix.

- <F1> b= [Enter], enter the elements of \underline{b} into cells G1:G3
- <I1> inverse= [Enter], Highlight J1:L3
type = MINVERSE(B1:D3) [Ctrl-Shft-Ent] *This gives A^{-1}*
- <N1> sol = [Enter], Highlight O1:O3
type = MMULT(J1:L3,G1:G3) [Ctrl-Shft-Ent] *This gives $A^{-1}\underline{b}$ which equals \underline{x}*

Hence write down the solutions $x =$ $y =$ $z =$

Task 4

Use the method of Task 3 to solve the following:

$$\begin{array}{rcl}
 & & x + 2y + z + u + v + w = 11 \\
 & & x - y - z + 2u + v = 3 \\
 & & 3x - 2y + u = 1 \\
 & & x + 2z + 3w = 9 \\
 & & z + u = 3 \\
 & & 2y + u + w = 8 \\
 (a) \quad & \begin{array}{l} x + 2y - z + w = 1 \\ x - y - 2z - w = -8 \\ 2x - y + 3z + w = 13 \\ -4x - 2y + 3z + w = 3 \end{array} & (b) \quad \begin{array}{l} x - y - z = 1 \\ -x + y + 2z = 0 \\ 2x - y + z = 6 \end{array} & (c) \quad \begin{array}{l} x + 2y + z + u + v + w = 11 \\ x - y - z + 2u + v = 3 \\ 3x - 2y + u = 1 \\ x + 2z + 3w = 9 \\ z + u = 3 \\ 2y + u + w = 8 \end{array}
 \end{array}$$

Write down the solutions:

$$\begin{array}{l}
 (a) \quad x = \quad y = \quad z = \quad w = \\
 (b) \quad x = \quad y = \quad z = \\
 (c) \quad x = \quad y = \quad z = \quad u = \quad v = \quad w =
 \end{array}$$

Bonus Task: To be tackled if you have time. (Please note: I constructed this problem some time ago when the prices were a bit more realistic! — OK)

One Friday lunchtime, a group of students went to a local pub for a (liquid) lunch. Unfortunately, only one of them had any money. But this student kindly offered to pay for them all on the condition that they refunded the money at a later time. During the course of this “lunch” four rounds were purchased, each consisting of combinations of bitter, lager, cider and stout. Due to an oversight the generous student made a record of who had drunk how much of each beer and what the total cost of each round was, but failed to make a record of how much was charged for a pint of each of the beers! Fortunately the student diligently paid attention in lectures and realised that it was possible to calculate the cost of the beers from the information available ...

The available data are summarised in the table below

Round no.	Pints of beer bought				Cost of Round
	Bitter	Lager	Cider	Stout	
1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	$\frac{1}{2}$	£8.99
2	$2\frac{1}{2}$	$\frac{1}{2}$	2	$2\frac{1}{2}$	£10.46
3	2	$2\frac{1}{2}$	3	1	£11.78
4	1	2	0	$2\frac{1}{2}$	£7.99

The student set the problem up in matrix form, denoting the costs of pints of bitter, lager, cider and stout as w , x , y and z respectively, giving

$$\begin{pmatrix} 1\frac{1}{2} & 2 & 2\frac{1}{2} & \frac{1}{2} \\ 2\frac{1}{2} & \frac{1}{2} & 2 & 2\frac{1}{2} \\ 2 & 2\frac{1}{2} & 3 & 1 \\ 1 & 2 & 0 & 2\frac{1}{2} \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8.99 \\ 10.46 \\ 11.78 \\ 7.99 \end{pmatrix}$$

The student was aware of the problems that could occur if the matrix was not invertible, and checked that the inverse could be found. Thus in full confidence the student proceeded to calculate the cost of each beer. The results were

A pint of bitter costs	£-3.86
A pint of lager costs	£0.60
A pint of cider costs	£4.58
A pint of stout costs	£4.26

Verify that these are the correct answers for the given data using Excel.

The student was puzzled! On checking the calculations no errors were found! The charges made by the publican were also correct, as were all the records of numbers of drinks and costs of rounds.

If nobody made any errors why does it look like the students were being paid to drink bitter? What were the correct costs of the drinks?

[Hint: If the price of a pint of beer is £1.99, how much does half a pint cost?]

Part 2

In this section we use Excel to numerically find the solution of nonlinear simultaneous equation. The self reference property and the iteration that follows can be applied to matrices, thus Newton's extended method is appropriate.

Task 5

Solve $z = f(x, y) = 2 - 4x^2 - 2xy^2 = 0$ and $z = g(x, y) = 4xy - y^2 - 1 = 0$

Before trying to solve these equations we need to know approximately where the solutions lie. Derive is capable of plotting curves, not only when y is expressed explicitly in terms of x as $y = f(x)$, but also when y is expressed implicitly in terms of x as in this problem.

Use Derive to plot the curves given by $2 - 4x^2 - 2xy^2 = 0$ and $4xy - y^2 - 1 = 0$ as follows:

(You will need to enter = 0 when inputting the expressions)

- Author the equations $2 - 4x^2 - 2xy^2 = 0$ and $4xy - y^2 - 1 = 0$
- Switch to the plot window and plot in the usual way.
- Write down approximations to the root (*1 decimal place is sufficient*) There should be three roots in total between $x = -1$ and $x = 1$

Hence: root₁ = (,) root₂ = (,) root₃ = (,)

The first step of the scheme is given by

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} - \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}_{(x_0, y_0)}^{-1} \begin{pmatrix} f \\ g \end{pmatrix}_{(x_0, y_0)}$$

thus we have to first obtain the partial derivatives of the two functions. This can be done by hand or you can use Derive, but first try finding these derivatives by hand.

Partial derivatives using Derive

This is carried out in the same way as for ordinary differentiation with the only difference that if the function has more than one variable Derive will give you the option of choosing which variable you wish to differentiate with respect to. For the above functions:

- author the function $2 - 4x^2 - 2xy^2$
- Select **Calculus Differentiate**
- Choose the variable to be x , select **Simplify**
You will now have $f_x(x, y)$
- repeat the process but choose the variable to be y to give $f_y(x, y)$.
- author the function $4xy - y^2 - 1$ and repeat the above steps to obtain $g_x(x, y)$ and $g_y(x, y)$.

Write down the partial derivatives:

$$f_x(x, y) = \qquad f_y(x, y) = \qquad g_x(x, y) = \qquad g_y(x, y) =$$

Hence complete:

$$J = \begin{pmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}$$

Iteration using Excel

Start a new worksheet

The following guides you through the implementation of the iterative scheme with the starting values $x_0 = -0.7$ and $y_0 = -0.5$.

- Construct two *user defined functions* in Excel. To do this open a VBA module, as in the last worksheet. Enter the following functions for $f(x, y)$ and $g(x, y)$.

```
function f(x,y)
f=2-4*x^2-2*x*y^2
End function
```

```
function g(x,y)
g= 4*x*y-y^2-1
End function
```

- On the new worksheet enter the following
<A1> X0= [Enter] <B1> =-0.7 [Enter]
<A2> Y0= [Enter] <B2> =-0.5 [Enter]
<A3> Matrix J [Enter]
<A4> * [Enter] <B4> * [Enter]
<A5> * [Enter] <B5> * [Enter]

where * indicates the expressions for the four partial derivatives found above. Remember that on the sheet x is replaced by **B1** and y by **B2**.

```
<A6> Inverse J [Enter]
Highlight cells A7:B8
type =MINVERSE(A4:B5) [Ctrl-Shft-Ent]
```

<A10> f= [Enter] <B10> =f(B1,B2) [Enter]
 <A11> g= [Enter] <B11> =g(B1,B2) [Enter]

<A13> X1=[Enter] <A14> Y1=[Enter]

Highlight cells **B13:B14**

type =**B1:B2-MMULT(A7:B8,B10:B11)** [Ctrl-Shift-Ent]

At this point there is no self (circular) reference. To allow Excel to carry out self (circular) reference set the option as before, that is to say:

- Find menu for iterative calculations (see guide on web page if you have forgotten).
 - Tick **Enable iterative calculation** set **maximum iterations** equal to 1 and **maximum change** equal to 0.001
 - Additionally when doing the iteration with matrices, as here, you should gain complete control of when the calculations are carried out.
- To start the iteration you create a self reference by entering the following:

<B1> =B13 [Enter] <B2> =B14 [Enter]

Since we have selected manual no calculation will take place until you press the **F9** key. One step of the iteration will now take place. To carry out the iterations repeatedly press the **F9** key until the input values of x and y (cells B1 and B2) agree, to the desired accuracy, with the output values of x and y (cells B13 and B14).

Solution: $x = \dots\dots\dots$ $y = \dots\dots\dots$

To find the other two solutions enter the starting values in **B1** and **B2** followed by the **F9** key. Remember that after changing these cells to the new starting values you have to change them back to =**B13** and =**B14** and repeatedly press the **F9** key to carry out the iteration.

Task 6

Solve the following problems as in Task 5, first plotting the functions using Derive to locate and identify approximations to the solutions.

Warning: Since you have set the calculation option to manual your spreadsheet will only update its entries on pressing the **F9** key.

(a) $4x^2 - y^2(2x + y) = 0$ $y^4 - 4x^3 = 0$ (in addition to (0,0) this has two solutions)

(b) $x \cos y - y \cos x = 0$ and $1 - x^2 - y^2 = 0$ (this has two solutions)

(c) $xy - y^2 - z^2 = 0$ $x \sin(4\pi y) + 8zx - 1 = 0$ $32xyz - 1 = 0$

In this problem it is not possible to plot the curves as there are too many variables. Use the method extended to three variables to find the solution near to $x_0 = 0.2$, $y_0 = 0.2$ and $z_0 = 0.2$. Can you locate any other solutions by changing the starting values?

In the worksheet 4π is entered as 4*pi()
and in the VBA module as 4*WorksheetFunction.pi()

