

Computational Mathematics/Information Technology

Worksheet 5

Part 1: Polynomial Fitting

Problem 1

Given the following data use **Excel and the method below** to construct a unique polynomial $p(x)$, of degree at most five, through the six points. Hence plot the data and the polynomial on the same graph.

$x =$	0	0.1	0.2	0.4	0.5	0.7
$y =$	0	0.0841	0.1819	-0.3027	-0.4795	0.4599

Use your polynomial to estimate y at $x = 0.25$ and $x = 1$.

Use Derive to compute the integral of $p(x)$ over the data range from 0 to 0.7.

Procedure for above

- Set up the following column headers so that you can keep track of the contents of your spreadsheet. In row1 columns A to H enter the symbols: x, y, a0, a1, a2, a3, a4, a5.
- Enter the x values in cells A3 to A8 and the y values in cells B3 to B8.
- In cells C3 to H8 we now form the matrix of coefficients of the unknown a values. From the notes the coefficient of a_0 is always 1, the coefficient of a_1 is x , of a_2 it is x^2 etc.

Enter:

=1 in cell C3 and copy down ¹ to C8	<i>coefficients of a_0</i>
=A3 in cell D3 and copy down to D8	<i>coefficients of a_1</i>
=A3^2 in cell E3 and copy down to E8	<i>coefficients of a_2</i>
=A3^3 in cell F3 and copy down to F8	<i>coefficients of a_3</i>
=A3^4 in cell G3 and copy down to G8	<i>coefficients of a_4</i>
=A3^5 in cell H3 and copy down to H8	<i>coefficients of a_5</i>

- Before evaluating the a s we will label the cells next to where they will appear. In cells A10 to A15 enter a0=, a1=, . . . , a5=.
- To calculate the a values in cells B10 to B15 we need to calculate the product of the inverse of the matrix in cells C3:H8 and the column of y data values in cells B3:B8. (*see lecture notes.*) We do this using the array functions MMULT and MINVERSE as follows:

Highlight B10 to B15

Enter =MMULT(MINVERSE(C3:H8),B3:B8) **Ctrl-Shift-Enter**.

B10 to B15 should now contain the values of a_0 to a_5 respectively.

- Write out the polynomial

$p(x) =$

- To plot the data and the polynomial we first need to create a list of (x, y) values for $p(x)$.
 - In cells A18 to A53 enter the x values from 0 to 0.7 in steps of 0.02. You can use the auto fill method by first entering 0 in A18 and 0.02 in cell A19. Highlight A18:A19 and then copy down by grabbing the + handle in the righthand bottom corner of A19 and drag and drop to A53.
 - To enter $p(x)$ in cells B18 to B53. In cell B18 enter the polynomial as:
 =\$B\$10+\$B\$11*A18+\$B\$12*A18^2+\$B\$13*A18^3 +\$B\$14*A18^4+\$B\$15*A18^5
 - Copy down B18 to B53.

¹use the mouse to grab the + in the righthand bottom corner of C3 and drag down to C8

- To plot the two sets of data on the same chart first plot the six given data points just as points:
 - Highlight the data A3:B8
 - Select Insert from the top row of options, then select Scatter from the Chart options.
 - Select Chart sub-type for just plotting points; the first in the list. This should give a chart with the six data points.
- To add the data for $p(x)$:
 - Highlight A18:B53 and select copy from the top Edit menu. (*or use the right mouse button or use Ctrl-C*)
 - Activate the outside boundary of the chart by left clicking just inside the chart.
 - Click on the arrow below Paste at the top left and select Paste Special. In the dialogue box select: New Series; Columns; Categories (X Values) in First Column; OK.
 - Since the Chart type is just points the data will be plotted as points. To change this so that the points are not shown but joined by straight lines you need to:
 - * Point to any data point on the chart, you should get a message when you do this, ensure that it says that you are pointing to series 2.
 - * Right click and select Change Series Chart Type from the pop up menu.
 - * Change the Chart sub-type to the one that joins points with straight lines but does not show the points, the last in the list. Select OK.

You should now have a chart showing the original six data points and a curve representing the polynomial that passes through the points. This polynomial can then be used to estimate the values of y between the given data points — **interpolation** — or values of y outside the range of the data points — **extrapolation**.

Calculate:

$$p(0.25) = \dots\dots\dots p(1.0) = \dots\dots\dots$$

To calculate $\int_0^{0.7} p(x) dx$, Author $a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5$ into **Derive** using the numerical values of the a s obtained above. Use Calculus Integrate with the definite integral option and the limits set equal to 0 and 0.7. Express your answer correct to three decimal places:

$$\int_0^{0.7} p(x) dx = \dots\dots\dots$$

Problem 1a(*Comparison with exact function*)

The above data is in fact generated by $f(x) = x \sin(10x)$. Use Derive to plot this and the polynomial $p(x)$. You can now see how good an approximation $p(x)$ is to $f(x)$ inside the data range. Outside the data range the approximation is poor and gets worse as x moves further out. You should note that the two curves, as expected, cross at the data points.

Problem 2 (*for the speedy*) The following problem illustrates how using a polynomial may not give a good result. Consider the data:

$x =$	-1	0	1	2	3	4
$y =$	-0.455	-0.833	-5.000	1.250	0.556	0.357

- Construct the polynomial of degree of at most five through the above points.
- As in Problem 1 use Excel to plot the data points and the polynomial on the same chart.
- Use Derive to plot the polynomial.
- The above data is in fact generated from $f(x) = \frac{1}{x - 1.2}$. Plot this in Derive on the same graph as $p(x)$. We can now observe the difference between $f(x)$ and $p(x)$ in the data range. They agree at the data points but have very little else in common!

Part 2: Linear Spline fitting

Problem 3 (Construct a Linear Spline)

Given the data points $\{(x_k, y_k) : k = 0, \dots, n\}$, with $x_0 < x_1 < \dots < x_n$ and $y_k = f(x_k)$, we wish to approximate $f(x)$ by joining consecutive data points with linear polynomials (straight lines) i.e. linear splines.

The complete spline on $[x_0, x_n]$ is denoted by $S(x)$. On each sub-interval we denote $S(x)$ by $s_k(x)$, $x \in [x_k, x_{k+1}]$, $k = 0, 1, \dots, n - 1$.

The formula for the linear spline can be written down explicitly as

$$s_k(x) = a_k + b_k(x - x_k) \quad k = 0, 1, \dots, n - 1,$$

where in terms of x_k, x_{k+1}, y_k and y_{k+1} the coefficients

$$a_k = \dots \quad \text{and} \quad b_k = \dots$$

Given the following table of values,

x_k	0	1	2	3	4
$y_k = f(x_k)$	6.00000	4.62091	1.75156	0.03002	1.03907

and using either a calculator, Derive or Excel calculate the following:

$$\begin{aligned} a_0 &= \dots & b_0 &= \dots \\ a_1 &= \dots & b_1 &= \dots \\ a_2 &= \dots & b_2 &= \dots \\ a_3 &= \dots & b_3 &= \dots \end{aligned}$$

Hence write down,

$$S(x) = \begin{cases} s_0(x) = \dots & x \in [\quad , \quad] \\ s_1(x) = \dots & x \in [\quad , \quad] \\ s_2(x) = \dots & x \in [\quad , \quad] \\ s_3(x) = \dots & x \in [\quad , \quad] \end{cases}$$

In order to enter $S(x)$ as a single function into Excel so that it can be plotted we need to use “a compounded worksheet if statement” or a VBA programme containing the “if..then...else” structure. The VBA approach is clearer and therefore preferable, Problem 4 constructs such a VBA function and asks you to plot both $S(x)$ and the given data points.

Problem 4 (plotting and evaluating a spline) Since by choosing the correct chart type Excel is already capable of plotting points and then joining them with straight line segments, this problem to this extent is redundant. However it does show how to construct a user function that is defined in a piece-wise fashion over several intervals and how to use it to produce values and draw a graph.

- Start a new VBA module and complete the following to produce the spline function $S(x)$ above:

```

Function S(x as double)
if x>=0 AND x<=1 then
S=(your expression for s0(x) obtained above)
elseif x>=1 AND x<=2 then
:
else
S="out of range"
endif
End Function

```

- Start a fresh worksheet and enter the five pairs of data values in cells A1:B5.
- Plot a new chart for these data values using the chart wizard with the XY (Scatter) and points only subtype.
- To construct the spline for the data enter values of x in cells A8:A28 from $x = 0$ to $x = 4$ in steps of 0.2 (use the auto fill method as in Problem 1) In cell B8 type =S(A8) and then copy down to cell B28.
- Add this set of data for $S(x)$ to your chart. (Use the same method as in problem 1, remember to make sure that you also change the chart subtype for this set of data from just points, to join with straight lines.)

Problem 5 (Comparison with the actual function.)

One of the applications of using splines or polynomials is to replace a difficult known function with a simpler one. The data given in Problem 3 actually came from the function $f(x) = 3(\cos x + 1)$.

To add the graph of $f(x)$ to our chart in Problem 4:

- In cell C8 enter =3*(cos(A8)+1) and copy down to cell C28.
- Highlight the values of x in cells A8:A28, then hold down the control key and highlight the $f(x)$ values in cells C8:C28. You should now have entries in the A and C columns selected, but not the B column. Copy using the Edit menu or use the right-mouse button.
- The data is now saved and ready to be pasted onto the chart. Again use the method of Problem 1 with Paste Special selected from the Edit menu.
- Change the chart type for this data from points only to join by lines using the method in Problem 1.

The graph indicates how good an approximation the spline is to $f(x)$. We may be interested in the error between the spline and $f(x)$ at given values of x .

Calculate:

$$|f(0.5) - S(0.5)| = \dots\dots\dots, \quad |f(1.5) - S(1.5)| = \dots\dots\dots$$

(In Excel use the worksheet function =ABS to calculate | |)

Problem 6 (optional for the really dedicated)

Use Derive or otherwise to calculate:

$$I_0 = \int_0^4 S(x) dx \quad \text{and} \quad I_1 = \int_0^4 f(x) dx$$

Hence complete:

$$I_0 = \dots\dots\dots \quad I_1 = \dots\dots\dots \quad |I_0 - I_1| = \dots\dots\dots$$

Assessment Exercises Worksheet 5

Surname:.....

Forename.....

Group No².



This sheet is to be handed in at the lab session(s) associated with Worksheet 5

1. For Problem 1 on Worksheet 5 write down the following:

(a) $p(x) =$

(b) $\int_0^{0.7} p(x) dx =$

(c) $p(0.25) =$ $p(1.0) =$

2. For Problem 3 on Worksheet 5 write out the complete linear spline $S(x)$ with its intervals of definition:

$$S(x) = \begin{cases} S_0(x) = \dots\dots\dots & x \in [\quad , \quad] \\ S_1(x) = \dots\dots\dots & x \in [\quad , \quad] \\ S_2(x) = \dots\dots\dots & x \in [\quad , \quad] \\ S_3(x) = \dots\dots\dots & x \in [\quad , \quad] \end{cases}$$

3. For Problem 5 on Worksheet 5:

(a) $|f(0.5) - S(0.5)| =$

(b) $|f(1.5) - S(1.5)| =$

²Actuarial Science Students only