

# Computational Mathematics/Information Technology

## Worksheet 7

By now you should not need as many instructions, you will have seen most of the things you have to do before. If you are having difficulties refer back to previous worksheets. For example for iteration you can look at week 2 and matrices week 3.

### Problem 1

Show that the matrix  $M$  given by:

$$M = \begin{pmatrix} 0.5 & 0.2 & 0.3 \\ 0.2 & 0.2 & 0.6 \\ 0.3 & 0.6 & 0.1 \end{pmatrix}$$

is a Markov matrix.

- (i) If  $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$  has column sum  $\alpha$ , that is to say  $a_1 + a_2 + a_3 = \alpha$ , show that  $M\mathbf{a}$  also has column sum  $\alpha$ .
- (ii) Show that  $M \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ .
- (iii) If  $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$  calculate  $M\mathbf{v}$  and  $M^2\mathbf{v}$ .

What would you expect  $M^n\mathbf{v}$  to tend to as  $n$  increases?

### Problem 2

Given

$$M = \begin{pmatrix} \frac{1}{4} & \frac{1}{10} \\ \frac{3}{4} & \frac{9}{10} \end{pmatrix}$$

- (a) Use MMULT(M,M), where M denotes the range containing  $M$ , to calculate  $M^2$ . If  $M^2$  is in cell range A use MMULT(A,A) to obtain  $M^4$ .
- (b) Evaluate  $M^4\mathbf{x}_0$  where  $\mathbf{x}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
- (c) Using Excel and the iterative scheme  $\mathbf{x}_{k+1} = M\mathbf{x}_k$ :
- Calculate  $M^4\mathbf{x}_0$  and compare your result with (b).
  - Calculate  $M^n\mathbf{x}_0$  as  $n$  tend to infinity.

Refer back to previous worksheets for guidance about finding the limit of large  $n$  of quantities like  $M^n\mathbf{x}$ .

**Problem 3**

In the weather forecast problem in lectures the question was posed “What is the probability of the weather being sunny in 4 days time if today is sunny?”.

(i) Use Excel or Derive to calculate

$$M^4 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

where  $M$  is the Markov matrix for this problem from lectures.

The probability it is sunny is .....

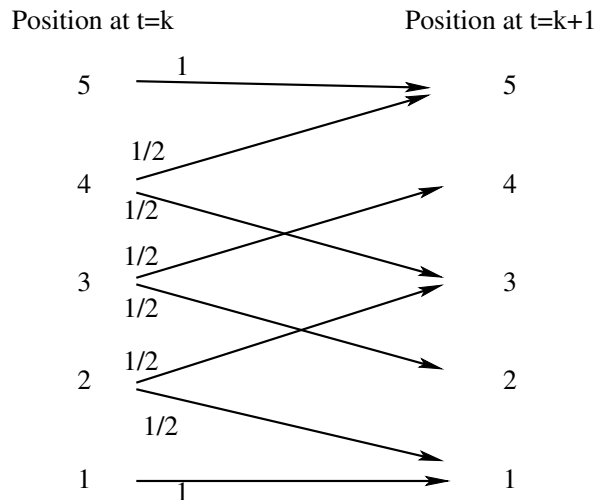
(ii) Answer this by calculating an exact answer using the representation of the matrix  $M$  as  $M = RDR^{-1}$  with  $D$  a diagonal matrix. (I.e., write the answer as a single expression involving whole numbers and fractions.)

The probability it is sunny is .....

Verify your answers are the same.

**Problem 4**

In the problem from lectures with the transition diagram



You win if you get to position 5 and lose if you get to position 1.

Use excel to calculate the limit of

$$M^n \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

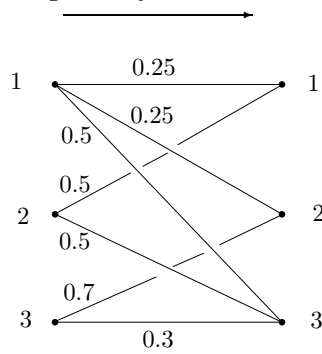
as  $n \rightarrow \infty$  where  $M$  is the matrix corresponding to this transition diagram.

What is the probability of winning the game if you start in position 2? .....

What is the simple argument from lectures that would have given you this answer in a couple of sentences?

**Problem 5**

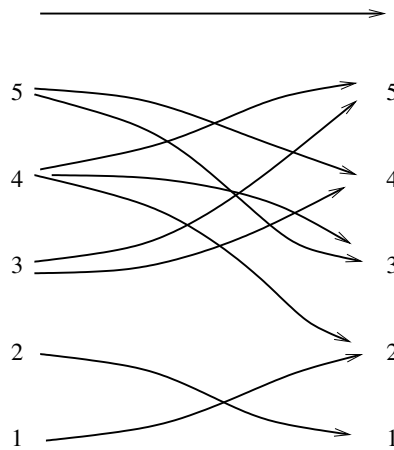
Write down the Markov matrix  $M$  given by the transition diagram



Find the limit of  $M^n \begin{pmatrix} 0.5 \\ 0.25 \\ 0.25 \end{pmatrix}$  as  $n$  gets large.  $\begin{pmatrix} \dots\dots\dots \\ \dots\dots\dots \\ \dots\dots\dots \end{pmatrix}$

**Problem 6**

The following transition diagram describes a small world wide web:



Confirm that for a “random surfer” the probability of ending up on a given page is independent of the starting position when  $0 \leq \alpha < 1$  (see lecture notes for explanation).

Note: The convergence when  $\alpha$  is near 1 is slow, and for  $\alpha = 1$  it does not converge. Confirm that for  $\alpha = 1$  the random surfer always ends up in pages 1 or 2.

Find the PageRanks of the pages for  $\alpha = 0.8$ .

Page 1: ....., Page 2: ....., Page 3: .....,

Page 4: ....., Page 5: .....

The effect of some well placed links can be significant. Consider what happens if there is one additional link added from Page 2 to Page 4. Find the PageRanks of the pages for  $\alpha = 0.8$  with this extra link.

Page 1: ....., Page 2: ....., Page 3: .....,

Page 4: ....., Page 5: .....

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<sup>1</sup>Discuss

# Assessment Exercises Worksheet 7

## Markov Chains and Google

Surname:.....

Forename:.....

Group No<sup>2</sup>.

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*For the Actuarial Science students this sheet is to be handed in at the associated lab session. For Maths students it should be handed in at the second of the two associated lab sessions. Late submission is only allowed with the permission of the lab tutor.*

*Where appropriate express answers correct to four decimal places*

- For Problem 3 write down the two answers:
  - Solution from Excel: .....
  - Exact solution: .....
- For Problem 4 what is the probability of winning from position 2? .....  
Give the simple argument given in lectures that gives this answer:

3. For problem 5 give the limit of  $M^n \begin{pmatrix} 0.5 \\ 0.25 \\ 0.25 \end{pmatrix}$  as  $n$  gets large.  $\begin{pmatrix} \dots\dots\dots \\ \dots\dots\dots \\ \dots\dots\dots \end{pmatrix}$

4. From Problem 6 write down the PageRanks for  $\alpha = 0.8$   
Page 1: ....., Page 2: ....., Page 3: .....,  
Page 4: ....., Page 5: .....

Give the PageRanks with the extra link:

Page 1: ....., Page 2: ....., Page 3: .....,  
Page 4: ....., Page 5: .....

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<sup>2</sup>Actuarial Science Students only