Computational Mathematics/Information Technology

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This course looks simultaneously at mathematically based problems and associated pieces of software.

We will look at some mathematics topic and use software used to solve specific problems.

Object — to extend your mathematical knowledge, and learn to use appropriate pieces of software which will reinforce the mathematics.

The course consists of two distinct parts:

- **Theory** This contained in the lectures which are for one session per week over twenty weeks.
- **Computer Practicals** The use of the software and development of the mathematical ideas are dealt with in the computational mathematics computer laboratories. These laboratories are scheduled for two hours per fortnight for each Actuarial Science student and one hour per week for the Mathematics students.

The following pieces of software are (probably) considered:

- **Derive**
- **Excel**
- **Minitab**
- **Google**

There is a set of notes produced by Dr Graham Bowtell which is available on the web, and will be available as a hard copy shortly.

Important Note — Not all the material in these notes will be used on the course and as such the notes should be considered as a reference for the lectures and labs and not a replacement. I may also decide to do different things not covered in the notes.

Sketching Curves

The object of this section is to understand some properties of functions and use these to sketch curves and interpret such sketches.

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Unless otherwise stated all variables and all functions are real valued.

To sketch the curve of a given function $y = f(x)$ the following features need to be identified.

- Where is $f(x)$ continuous
- Intervals on which $f(x)$ is increasing or decreasing
- Stationary points
- Local maxima and minima
- Points of inflection
- x and y intercepts
- Asymptotes

Basic idea — can you draw it without taking your pen off the paper?

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A function is continuous at a point x_0 if

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Note: This must be true for x approaching x_0 from the left *and* right.

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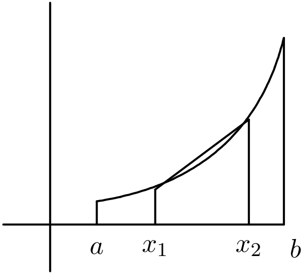
If $f(x)$ is continuous for all x then we say '$f(x)$ is continuous'.

Increasing and decreasing functions

Basic idea — does the curve go up or down?



We make the following definition:

$f(x)$ is an **increasing function** on the interval $[a, b]$ if for all x_1 and $x_2 \in [a, b]$

$$x_2 > x_1 \Rightarrow f(x_2) > f(x_1)$$

Just a mathematical way of saying 'if $f(x)$ is increasing, then as x increases so does $f(x)$.

Note:

This has nothing to do with having a positive gradient!

However, if $f(x)$ is differentiable then we can make the following deduction:

Clearly

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0$$

which, since as this tends to $f'(x_1)$ as $x_2 \to x_1$, implies that $f'(x_1) \geq 0$.

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which, since as this tends to $f'(x_1)$ as $x_2 \to x_1$, implies that $f'(x_1) \geq 0$.

Note: **Not** $f'(x_1) > 0$ — think of $f(x) = x^3$ at $x_1 = 0$

If $f(x)$ is differentiable for all x in the interval $[a, b]$ we can make the following statements:

$$f(x) \text{ increasing on } [a, b] \Rightarrow f'(x) \geq 0 \text{ on } [a, b]$$

If $f(x)$ is differentiable for all x in the interval $[a, b]$ we can make the following statements:

$$f(x) \text{ increasing on } [a, b] \Rightarrow f'(x) \geq 0 \text{ on } [a, b]$$

$$f'(x) > 0 \text{ on } [a, b] \Rightarrow f(x) \text{ increasing on } [a, b]$$

Similarly, if $f(x)$ is differentiable for all x in the interval $[a, b]$ then

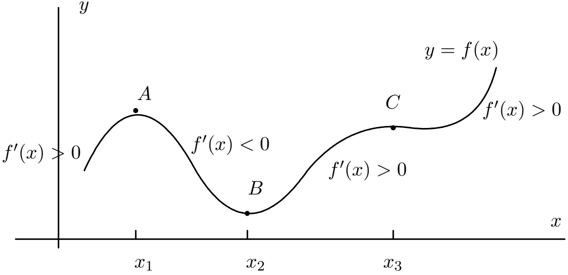
$$f(x) \text{ decreasing on } [a, b] \Rightarrow f'(x) \leq 0 \text{ on } [a, b]$$

$$f'(x) < 0 \text{ on } [a, b] \Rightarrow f(x) \text{ decreasing on } [a, b]$$

Stationary points

Basic idea — points when the gradient is zero.

The point $(x, f(x))$ is a **stationary point** of $f(x)$ if $f'(x) = 0$.



Here A, B and C are stationary points.

Note: at A and B the curve *turns*. This leads to the following definition:

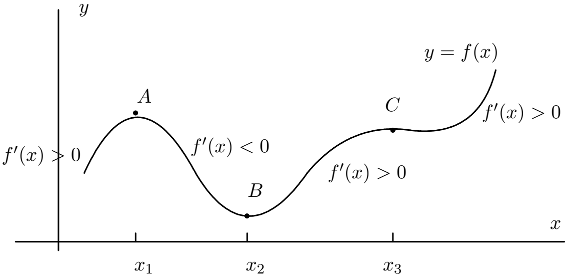
- A point about which $f'(x)$ changes sign is called a **turning point**. Clearly at such a point $f'(x) = 0$, provided f' is continuous at the given point. I.e., the point is stationary.

It is also clear that in the locality of the two points A and B, $f(x)$ is maximal and minimal respectively.

Local maximums and minumums are defined by

- $f(x)$ has a **local maximum** at x_1 if, for all $x \neq x_1$ in the neighbourhood of x_1, $f(x) < f(x_1)$
- $f(x)$ has a **local minimum** at x_2 if, for all $x \neq x_2$ in the neighbourhood of x_2, $f(x) > f(x_2)$

Classification of stationary points



In the diagram these are the three stationary points: A, B and C. We have seen that A and B are a local maximum and minimum respectively. The point C is a point of inflection, to be defined later.

The following criteria allows us to classify stationary points by considering either $f'(x)$ on either side of a stationary point or $f''(x)$ at the stationary point.

- If to the left of the stationary point $f'(x)$ is positive and to the right of the stationary point $f'(x)$ is negative then the stationary point is a **local maximum**.

The following criteria allows us to classify stationary points by considering either $f'(x)$ on either side of a stationary point or $f''(x)$ at the stationary point.

- If to the left of the stationary point $f'(x)$ is positive and to the right of the stationary point $f'(x)$ is negative then the stationary point is a **local maximum**.
- An alternative test for a local maximum is to observe that at A in the figure $f'(x)$ varies from being positive to the left of x_1 to being negative to the right of x_1, thus $f'(x)$ is decreasing and hence its derivative (if it exists) is negative. I.e., $f''(x) < 0$. Thus we can say that

  If $f'(x_1) = 0$ and $f''(x_1) < 0$ then $f(x)$ has a local maximum at x_1

- If to the left of the stationary point $f'(x)$ is negative and to the right of the stationary point $f'(x)$ if positive, as at the point B, then the stationary point is a **local minimum**.
- An alternative test for a local minimum is to observe that at B $f'(x)$ varies from being negative to the left of x_2 to being positive to the right of x_2, thus $f'(x)$ is increasing and hence its derivative is positive. I.e., $f''(x) > 0$. Thus we can say that

  If $f'(x_2) = 0$ and $f''(x_2) > 0$ then $f(x)$ has a local minimum at x_2

- If $f'(x)$ has the same sign immediately to the left and right of the stationary point, as at the point C, the stationary point is an example of a point of inflection (defined below).

Example

Find the stationary points of $f(x) = x^4 - 6x^2 + 8x$ and hence classify them as local maxima, local minima or point of inflection.

Point of inflection

A point of inflection is any point on the curve where its second derivative changes sign.

Note: the point does not necessarily have to be a stationary point

Formally we have:

- If the sign of $f''(x)$ immediately to the left of $x = c$ is different to the sign of $f''(x)$ immediately to the right of $x = c$ then the curve $y = f(x)$ has a point of inflection at $x = c$. Clearly since $f''(x)$ is changing sign as x passes through c we can deduce that $f''(c) = 0$
- The following point should be noted: if $x = c$ is a point of inflection for $f(x)$ then $f''(c) = 0$. However the converse is not true. ie $f''(c) = 0$ does not imply that $f(x)$ has a point of inflection at $x = c$

A simple second derivative test $f''(x_3) = 0$ cannot be used to classify the point C as a point of inflection in

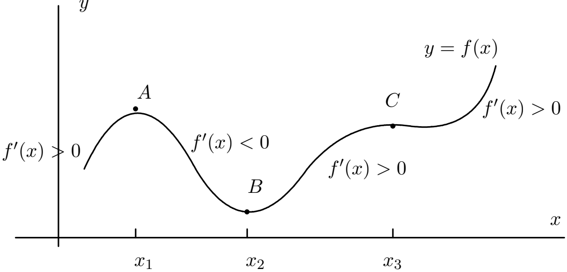


fig **??**

However we can see that it satisfies the definition of a point of inflection as follows:

To the left of $x = x_3$ the first derivative $f'(x)$ is decreasing to zero at $x = x_3$ which means that $f''(x) < 0$ to the left of $x = x_3$. To the right of $x = x_3$ the first derivative $f'(x)$ is increasing from zero at $x = x_3$, which means that $f''(x) > 0$ to the right of $x = x_3$. Thus $f''(x)$ changes sign at $x = x_3$ and hence $f(x)$ has a point of inflection at $x = x_3$ Thus for a stationary point where the second derivative vanishes we can check to see if its a point of inflection by looking at the sign of $f'(x)$ either side of the point.

- $f''(x) = 0$ is not sufficient to establish a point of inflection:

  In the definition $f''(x)$ needs to change sign for a point to be a point of inflection, and at such a point $f''(x) = 0$. However $f''(x) = 0$ is not sufficient to guarantee a point of inflection. Consider $y = x^4$, clearly this has a minimum at $x = 0$ since $x^4 \geq 0$ for all x. However $f'(0) = 0$ and $f''(0) = 0$. Thus $(0, 0)$ is a stationary point at which $f''(x) = 0$ but which is not a point of inflection.