Computational Mathematics/Information Technology

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Basic idea

We are going to look at a class of models from probability — **Markov Chains**.

These will use matrices and powers of matrices.

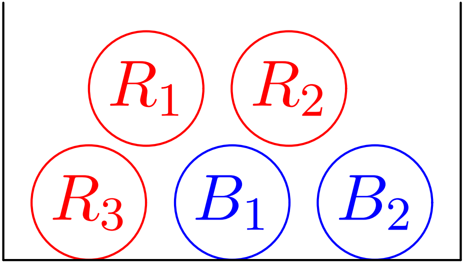
A typical problem: Days are either "sunny" or "not sunny". One of the better ways of predicting the weather for tomorrow is to say the weather will be just the same as today's.

Example: The probability of a correct forecast if today is sunny is $3/4$, and if today is not sunny is $2/3$. What is the probability of the weather being sunny in 4 days time if today is sunny? What is the probability in 100 days time?

This model has the important property — the **Markov property** — that tomorrow's weather only depends on today's weather. What happened yesterday is not important.

Conditional probability

Consider a bowl containing three red balls and two blue balls:



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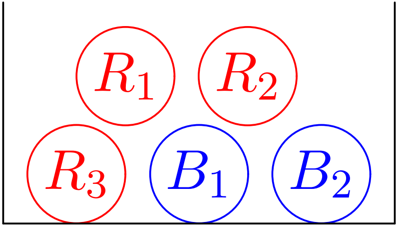
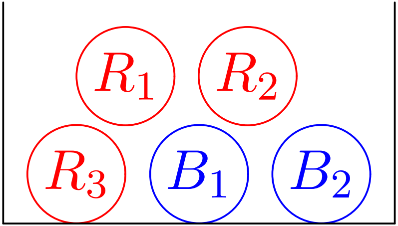
If balls are not replaced: If the balls are chosen at random, what is the the probability that the second ball is red?



OK — not the most difficult problem!

The answer is:



OK — not the most difficult problem!

The answer is: $3/5$.

But we will look at it also considering what happened with the first choice.

We can choose the first ball in five ways and, given the first choice, the second in four ways giving $5 \times 4 = 20$ possible ordered pairs of the form $R_1 B_1, \; B_2 R_2$, etc.

To calculate the number of pairs where the second choice is a red ball we could list all 20 possible pairs and then count how many ended in a red choice.

A more general approach would be to say that we have a success if we choose either a red followed by a red (RR) or a blue followed by a red (BR).

- For BR we can choose the first ball in 2 ways from 5 and the second in 3 ways from 4, giving a total of $2 \times 3 = 6$ ways from the 20 possible selections.
- Thus the probability of selecting BR,

$$P(BR) = \frac{2 \times 3}{20} = \frac{2}{5} \times \frac{3}{4} = P(B)P(R|B) = \frac{6}{20}$$

Where $P(B)$ is the probability of selecting a blue ball on the first draw and $P(R|B)$ is the probability of selecting a red ball on the second draw given that a blue ball was chosen on the first draw. $P(R|B)$ is referred to as the **conditional probability** of selecting a red ball given that a blue has already been chosen.

- For RR we can choose the first ball in 3 ways from 5 and the second in 2 ways from 4, giving a total of $3 \times 2 = 6$ ways from 20. As before we then have

$$P(RR) = \frac{3}{5} \times \frac{2}{4} = P(R)P(R|R) = \frac{6}{20}$$

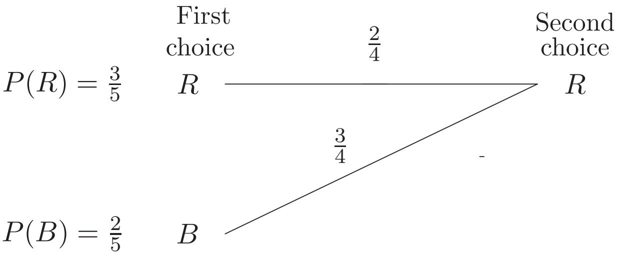
- Since to succeed only one or other of these can happen the total chance is obtained by adding the two, thus:

$$P(\text{Red Second}) = P(B)P(R|B) + P(R)P(R|R)$$

$$= \frac{6}{20} + \frac{6}{20} = \frac{12}{20} = \frac{3}{5}$$

Not a great surprise there!

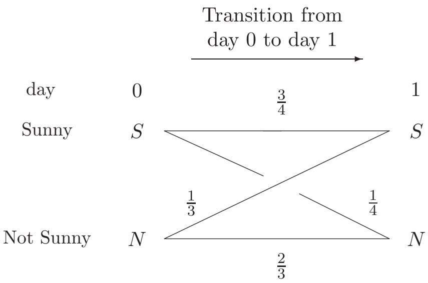
The above example can be represented by the following transition diagram which will then be developed into a Markov Chain shortly.



Markov Chain

Using the prediction that the weather for tomorrow will be just the same as today's:

The probability of a correct forecast if today is sunny is 3/4, and if today is not sunny is 2/3. This gives the following transition diagram:



Denoting the probability of being sunny on day k by $P_k(S)$ and not sunny by $P_k(N)$ we deduce from the diagram that:

$$P_1(S) = \frac{3}{4}P_0(S) + \frac{1}{3}P_0(N)$$

and

$$P_1(N) = \frac{1}{4}P_0(S) + \frac{2}{3}P_0(N)$$

We can write this in matrix form:

$$\begin{pmatrix} P_1(S) \\ \\ P_1(N) \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & \frac{1}{3} \\ \\ \frac{1}{4} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} P_0(S) \\ \\ P_0(N) \end{pmatrix}$$

which is denoted as $$\mathbf{P}_1 = M\mathbf{P}_0$$

Moving from day to day we have:

$$\mathbf{P}_1 = M\mathbf{P}_0 \quad \rightarrow \quad \mathbf{P}_2 = M\mathbf{P}_1 = M^2\mathbf{P}_0 \quad \ldots \quad \mathbf{P}_n = M^n\mathbf{P_0}$$

Calculating M^n for general n is clearly a difficult problem if M has many rows and columns.

Sometimes it is possible...

We can write the array M in the form

$$M = RDR^{-1}$$

where R is an invertible matrix and D is a diagonal matrix:

$$\begin{pmatrix} \frac{3}{4} & \frac{1}{3} \\ \frac{1}{4} & \frac{2}{3} \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 5/12 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 3 & -1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 4 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 5/12 \end{pmatrix} \begin{pmatrix} 1/7 & 1/7 \\ 3/7 & -4/7 \end{pmatrix}$$

We can see that

$$M^2 = \left(RDR^{-1}\right)\left(RDR^{-1}\right) = RD\left(R^{-1}R\right)DR^{-1}$$
$$= RDIDR^{-1} = RDDR^{-1} = RD^2R^{-1}$$

In a similar way we can show that

$$M^n = RD^nR^{-1}$$

and in general for a diagonal matrix

$$D^n = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^n = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}$$

and so

$$M^n = \begin{pmatrix} 4 & 1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & (5/12)^n \end{pmatrix}\begin{pmatrix} 1/7 & 1/7 \\ 3/7 & -4/7 \end{pmatrix}$$

It is straight forward to calculate M^4 now, and even easier to see what happens a long time in the future.

When $n = 100$ it is clear that $(5/12)^{100}$ is very small (around 10^{-38}). So

$$M^{100} \approx \begin{pmatrix} 4 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/7 & 1/7 \\ 3/7 & -4/7 \end{pmatrix} = \begin{pmatrix} 4/7 & 4/7 \\ 3/7 & 3/7 \end{pmatrix}$$

and whatever the weather is today the probability of it being sunny is $4/7$ and not sunny is $3/7$.

This last bit will be covered in Linear Algebra in Part 2 — so is beyond the scope of this course and you will not be expected to find R and D for a given M.

But you can still be expected to find M^4 using Excel, or to see if M^n converged for large n using iteration similar to previous material in this course.

The above example only had two possible states, we can use these ideas for much larger systems...