Computational Mathematics/Information Technology

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Asymptotes

A curve C is an asymptote to $y = f(x)$ if as the point $(x, f(x))$ moves away from the origin the graph of $y = f(x)$ tends to C.

Essentially we have two cases:

(i) $x = a$ is an asymptote to $y = f(x)$ if as $x \to a$, $\quad |f(x)| \to \infty$.

(ii) As $x \to +\infty$ or as $x \to -\infty$ y tends to the curve C

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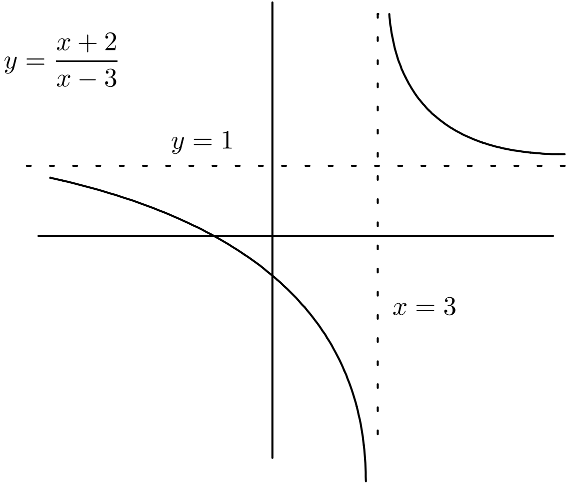
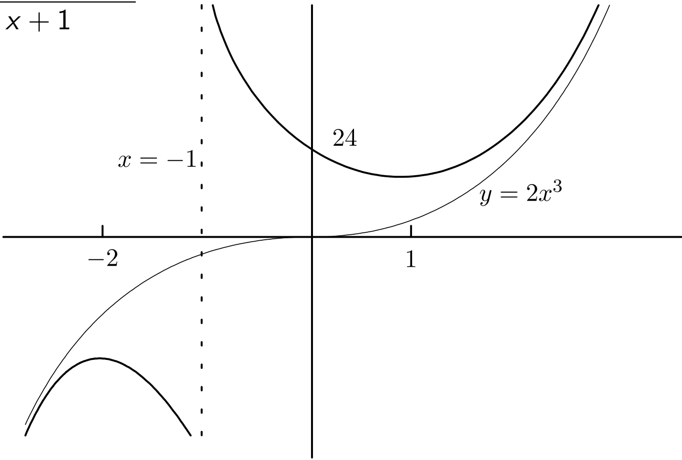
(ii) As $x \to +\infty$ or as $x \to -\infty$ y tends to the curve C

The curve C is quite often just a straight line but in general it can be any suitable curve


$y = \dfrac{x+2}{x-3}$
$y = 1$
$x = 3$

$$y = f(x) = \frac{2(x^4 + x^3 + 12)}{x + 1}$$



Simple asymptotes

Consider

$$y = \frac{x+2}{x-3} = f(x).$$

Clearly this function is not defined at $x = 3$ however:

- As $x \to 3$ from above $y \to +\infty$
- As $x \to 3$ from below $y \to -\infty$

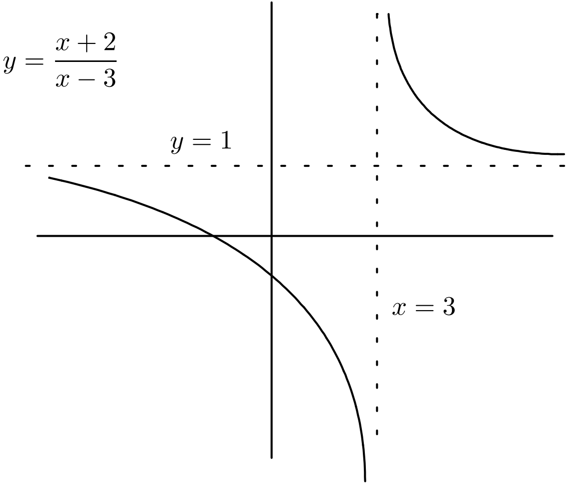
Thus $x = 3$ is a vertical asymptote.

To consider y as $x \to \infty$ we first rearrange $f(x)$ as

$$y = \frac{1 + \frac{2}{x}}{1 - \frac{3}{x}}$$

Clearly now as $x \to \pm\infty \quad y \to 1$.

Thus $y = 1$ is a horizontal asymptote.


$y = \frac{x+2}{x-3}$
$y = 1$
$x = 3$

Example

Sketch

$$y = f(x) = \frac{2(x^4 + x^3 + 12)}{x + 1}.$$

First look for asymptotes:

$$y = f(x) = \frac{2(x^4 + x^3 + 12)}{x + 1}.$$

Clearly this function is not defined at $x = -1$
However:

- As $x \to -1$ from above $y \to +\infty$
- As $x \to -1$ from below $y \to -\infty$,

(In both cases the numerator of $f(x)$ is positive in the neighbourhood of $x = -1$.)

Thus $x = -1$ is a vertical asymptote.

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Yes. To see this we rearrange $f(x)$ as follows:

$$y = 2x^3 + \frac{24}{x+1} \quad \text{clearly now as } x \to \pm\infty \quad y \to 2x^3$$

Thus $y = 2x^3$ is an asymptote for $y = f(x)$.

Next look for stationary points:

Using the rearranged form of $f(x)$ we see that

$$f'(x) = 6x^2 - \frac{24}{(x+1)^2}$$

Hence $f'(x) = 0$ implies that $x = 1$ or -2.

Note: $f(1) = 14$ and $f(-2) = -40$.

Look at the second derivatives:

$$f''(x) = 12x + \frac{48}{(x+1)^3}$$

Evaluating gives

$$f''(1) > 0 \quad \text{and} \quad f''(-2) < 0.$$

Hence we deduce that there is a local minimum at $x = 1$ and a local maximum at $x = -2$.

Look for points of infection:

$$f''(x) = 12x + \frac{48}{(x+1)^3} = 0$$

has no solutions (why?) so there are no points of inflection.

Look for intercepts:

Putting $x = 0$ into $f(x)$ we see that the y-intercept is given by $y = 24$.

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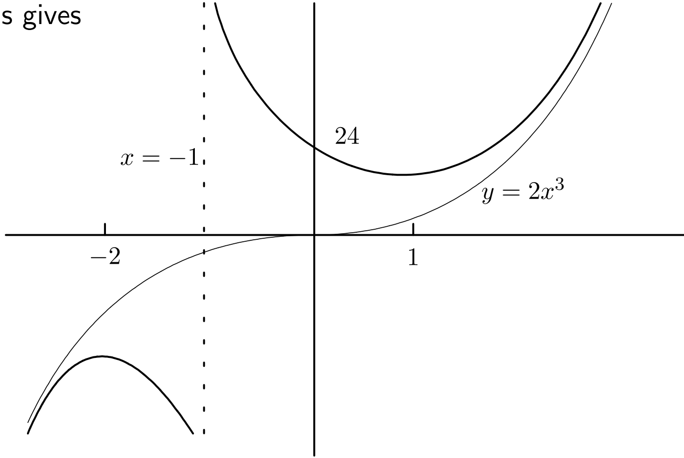
The solution of this problem is not trivial so will not be considered here.

Combining all this gives



A more accurate plot is