Computational Mathematics/Information Technology

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Finding roots — Introduction

Many problems require the solution of the equations of the form

$$f(x) = 0$$

Value of x that satisfy the equation are called **roots**.

We saw this previously when looking for points where the curve $y = f(x)$ cuts the x-axis.

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In general this is a difficult problem.

Quadratic Equations

$$ax^2 + bx + c = 0$$

has solutions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Cubic Equations

$$ax^3 + bx^2 + cx + d = 0$$

To find solutions let

$$q = \frac{c}{3a} - \frac{b^2}{9a^2}, \quad r = \frac{bc - 3da}{6a^2} - \frac{b^3}{27a^3}$$

Then let

$$s_1 = \left(r + (q^3 + r^2)^{1/2}\right)^{1/3}, \quad s_2 = \left(r - (q^3 + r^2)^{1/2}\right)^{1/3}$$

The roots are

$$z_1 = s_1 + s_2 - \frac{b}{3a}$$

$$z_2 = -\frac{s_1 + s_2}{2} - \frac{b}{3a} + \frac{i\sqrt{3}(s_1 - s_2)}{2}, \quad z_3 = -\frac{s_1 + s_2}{2} - \frac{b}{3a} - \frac{i\sqrt{3}(s_1 - s_2)}{2}$$

(Not examinable)

Quartic Equations

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

Find the real solution of

$$z^3 - bz^2/a + (bd - 4ae)z/a^2 - (ad^2 + b^2e - 4ace)/a^3 = 0$$

Then find the four roots of the two quadratic equations

$$w^2 + \left(\frac{b}{2a} \mp \left(\frac{b^2}{4a^2} + z - \frac{c}{a}\right)^{1/2}\right) w + \frac{z}{2} \mp \left(\left(\frac{z}{2}\right)^2 - \frac{e}{a}\right)^{1/2}$$

(Not examinable)

Quintic Equations and others

No general solution exists for quintic equations, or methods of writing down solutions for other problems except in special cases.

Higher dimensions

You may have several equations in several unknowns. For example

$$2x + y = 3, \quad x + 3y = 5.$$

Linear equations like this usually present very few problems (but not always).

A more difficult problem to solve is when the simultaneous equations are not linear, for example

$$f_1(x_1, x_2) = 2 - x_1^2 - x_2 = 0 \quad \text{and} \quad f_2(x_1, x_2) = 2x_1 - x_2^2 - 1 = 0.$$

Using vector notation can write sets of equations in a compact form: Set

$$\mathbf{x} = (x_1, x_2, \ldots, x_N) \quad \mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_N(\mathbf{x}))$$

Or more briefly:

$$\mathbf{f}(\mathbf{x}) = \mathbf{0}$$

where the zero vector $\mathbf{0} = (0, 0, \ldots, 0)$.

If we are asked to find a solution to

$$\mathbf{f(x) = 0}$$

We need to consider the following three questions.

- Does a solution exist? *existence*
- Is there one solution or many? *uniqueness*
- If using method of approximating the root how big is the error? *accuracy*

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It is not always easy to answer these question, particularly when we have more than 1 variable and 1 equation!

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Hence $x^2 + \cos 4x > 0$ for $|x| > 1$.

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It is also an even function, so if x is a root so is $-x$. Concentrate on $x \geq 0$

Calculate a few points:

$x =$	0	1/2	1
$f(x) = x^2 + \cos 4x$	1.00000	-0.16615	0.34636

We can use the fact that $x^2 + \cos 4x$ is continuous ...

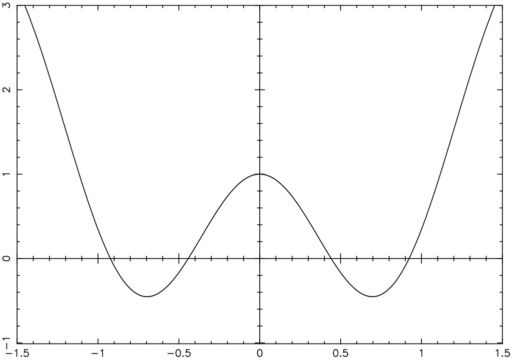
Theorem — a special case of the Intermediate Value Theorem

If $f(x)$ is continuous in the interval $a \leq x \leq b$ and $f(a) \times f(b) < 0$ then there exists at least one solution to $f(x) = 0$ in the interval $[a, b]$.

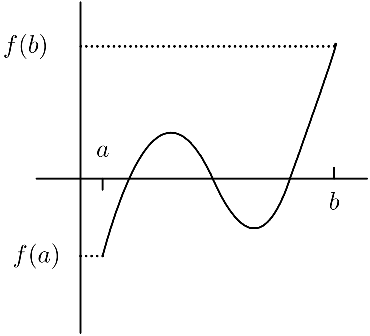
The theorem guarantees at least it one solution. However, there may be more than one solution in the interval $[a, b]$.

Here we have both $f(0) \times f(1/2) < 0$ and $f(1/2) \times f(1) < 0$, so we have at least one root in $[0, 1/2]$ and one root in $[1/2, 1]$.

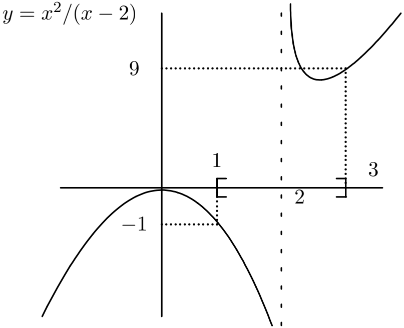
Graph of $y = x^2 + \cos 4x$

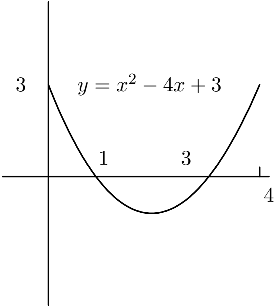


More than one solution:



The function must be continuous:





$f(1) \times f(3) < 0$, but no roots.

$f(0) \times f(4) > 0$, but has roots.

Having $f(a) \times f(b) < 0$ is not it essential for a root.

Finding a root — iterative schemes

In general we cannot find an exact representation of a root, so often we will find an approximate value. Usually this is done with an **iterative scheme**.

One such scheme follows the following steps:

- To solve $f(x) = 0$ we rewrite this equation in the form $x = g(x)$. Thus the roots of $f(x) = 0$ are the same as any value of x satisfying $x = g(x)$.
- To obtain the solution of $x = g(x)$ we take an initial first guess $x = x_0$ then calculate x_1 from the equation $x_1 = g(x_0)$. The process is then repeated using in general $x_n = g(x_{n-1})$ to generate the sequence $x_0, x_1, x_2, x_3, \ldots$
- If the sequence 'converges' to c as n tends to infinity then $x_n = g(x_{n-1})$ becomes $c = g(c)$. Thus the limit of the sequence satisfies $c = g(c)$ and thus $x = c$ is a solution of $f(x) = 0$.

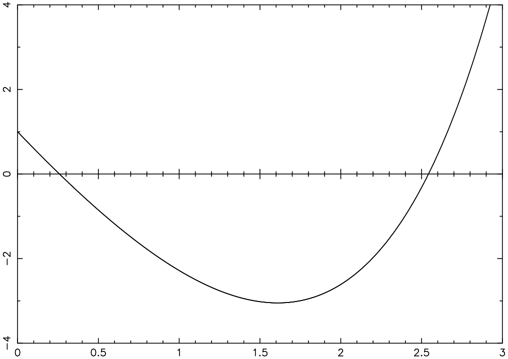
The value of c that satisfies $c = g(c)$ is called a **fixed point** of $g(x)$.

Definition - fixed point

Given a function $g(x)$ then $x = c$ is a **fixed point** of $g(x)$ if $c = g(c)$.

Example

Find the roots of $f(x) = e^x - 5x$.
Graph of $y = f(x)$:



First rewrite

$$f(x) = e^x - 5x = 0$$

Rewrite as

$$e^x = 5x \quad \text{or} \quad g(x) = e^x/5 = x.$$

Use iterative scheme

$$x_n = e^{x_{n-1}}/5.$$

$n =$	0	1	2	3	4	5	6	7	8
$x_n =$	1	0.544	0.344	0.282	0.265	0.261	0.260	0.259	0.259

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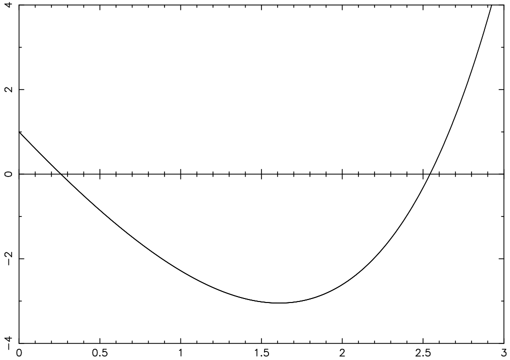
$x_n =$	3	4.017	11.11	13341	2×10^{5793}	...	...	...	...



We found the left root, but not the right root. What went wrong?

We can try another rearrangement:

$$f(x) = 0 \Rightarrow e^x - 5x = 0 \Rightarrow e^x = 5x \quad \Rightarrow \quad x = \ln(5x) = g(x)$$

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Using $x_n = \ln(5x_{n-1})$ with two starting values gives the following sequences

$n =$	0	1	2	3	4	5	6	7
$x_n =$	0.25	0.223	0.109	-0.602	*complex* ...	...	...	...
$x_n =$	1.0	1.609	2.085	2.344	2.461	2.510	2.530	2.538

The iterative scheme now works for the right root (slowly), but not the left one. Why?