Computational Mathematics/Information Technology

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2009–10

Last week we saw that one way to find roots of

$$f(x) = 0$$

is to rearrange it as

$$g(x) = x$$

and use the iterative scheme

$$x_n = g(x_{n-1})$$

Sometimes it converged to a root, sometimes it didn't.

A fixed point of

$$x_n = g(x_{n-1})$$

will be a point where the lines

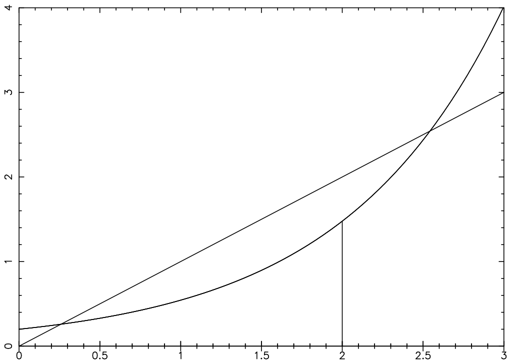
$$y = x \quad \text{and} \quad y = g(x)$$

cross.

We looked at the example of finding the roots of $f(x) = e^x - 5x = 0$ with two rearrangements:

1. $x = g(x) = e^x/5$
2. $x = g(x) = \ln(5x)$

Graph of $y = g(x) = e^x/5$ and $y = x$ with $x_0 = 2$:



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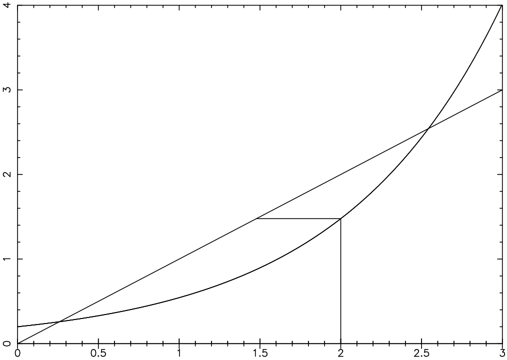

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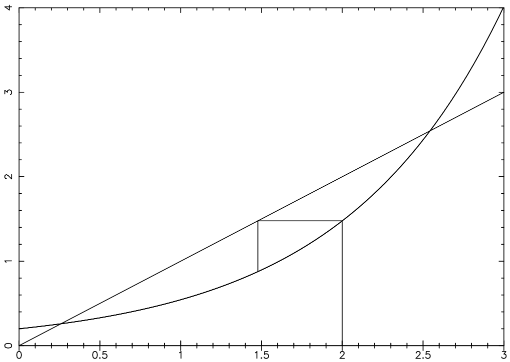

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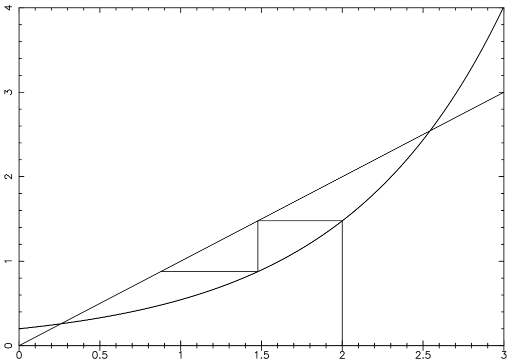

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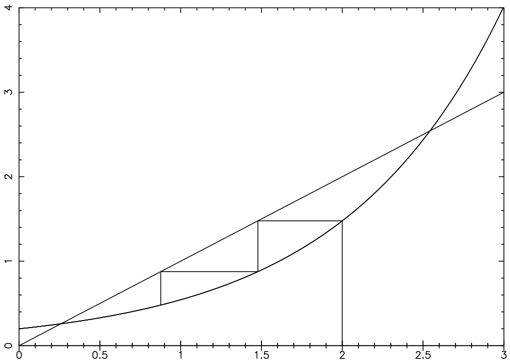

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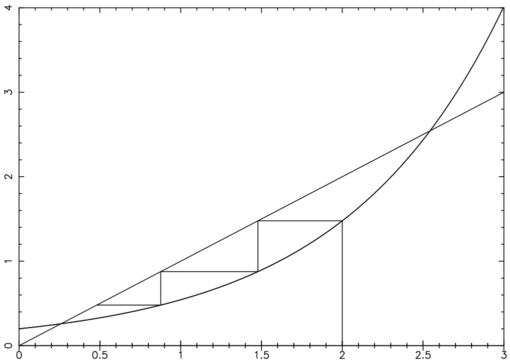

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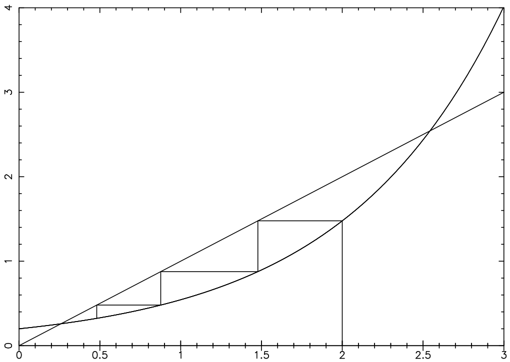

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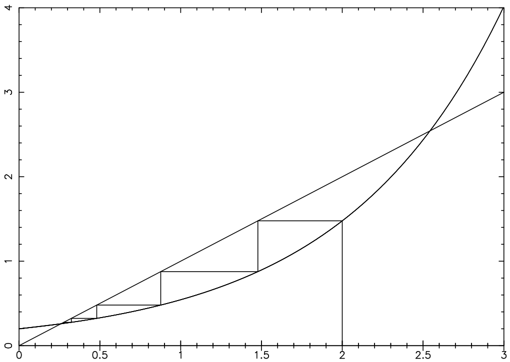
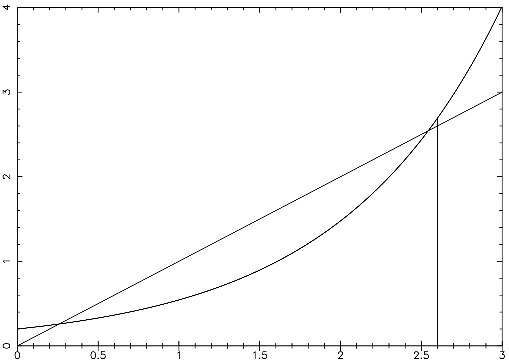


Graph of $y = g(x) = e^x/5$ and $y = x$ with $x_0 = 2.6$:



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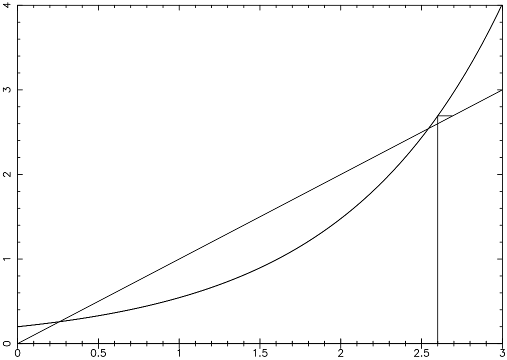

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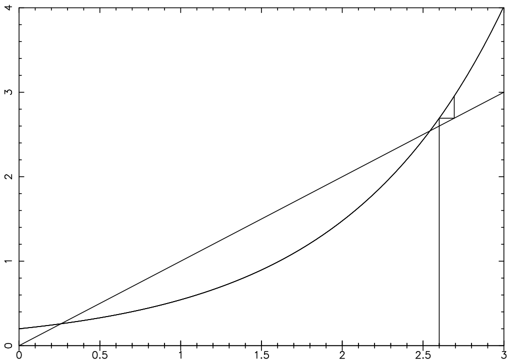

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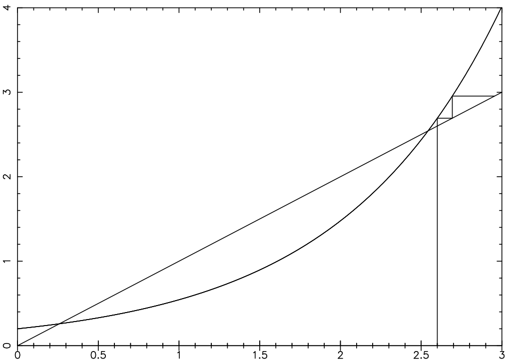

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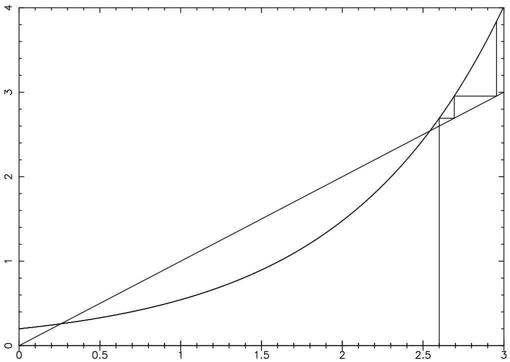

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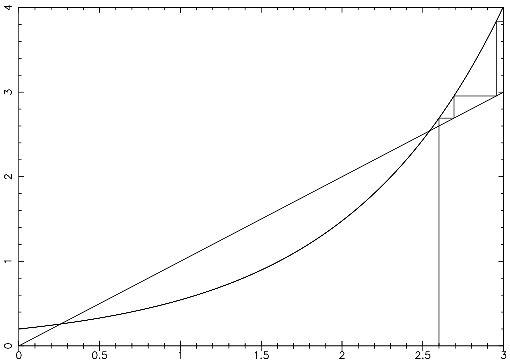

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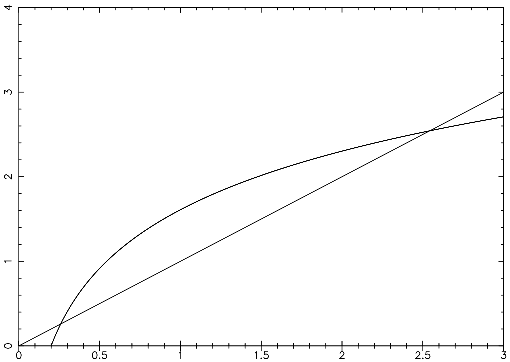
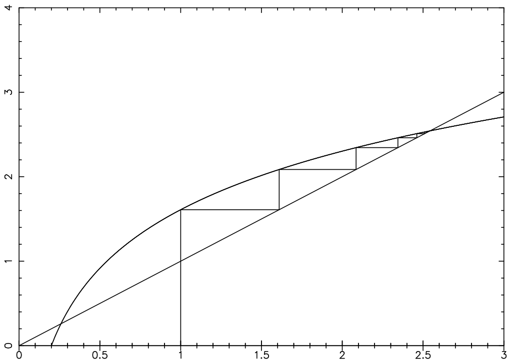


Graph of $y = g(x) = \ln(5x)$ and $y = x$ with $x_0 = 1$:



Convergence

If a root is x_c and $x_{n-1} = x_c + \epsilon$ then

$$x_n = g(x_{n-1}) = g(x_c+\epsilon) = g(x_c)+\epsilon g'(x_c)+\cdots = x_c+\epsilon g'(x_c)+\cdots$$

(Using Taylor series)

So approximate error is multiplied by $g'(x_c)$ after each iteration.

This method only converges if at a root x_c if we have

$$|g'(x_c)| < 1$$

This method has the advantage that it is very easy to set up.

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This method has the disadvantage it doesn't always work or give all of the roots.

Newton's method

Here we use Newton's method

$$x_n = x_{n-1} - \frac{f\left(x_{n-1}\right)}{f'\left(x_{n-1}\right)}$$

for finding root of an equation $f(x) = 0$.

Example

$$f(x) = \frac{Vx^2 - 1}{x^2}$$

with $V = 2$ and $x_0 = 1$.

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$$f'(x) = \frac{2Vx.x^2 - (Vx^2 - 1).2x}{x^4}$$

$$\left[\text{using } \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}\right]$$

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Or more simply

$$f(x) = V - 1/x^2 \qquad \text{so} \qquad f'(x) = 2/x^3$$

Scheme is

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} = x_{n-1} - \frac{1}{2}(Vx_{n-1}^2 - 1)x_{n-1}$$

Scheme is

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} = x_{n-1} - \frac{1}{2}(Vx_{n-1}^2 - 1)x_{n-1}$$

Note that the root of

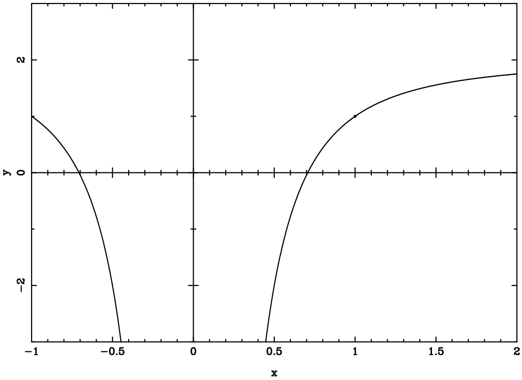
$$f(x) = (Vx^2 - 1)/x^2 = 0$$

is

$$1/\sqrt{V}$$

In this case Newton's method does not involve any division.
This has been used as a fast way of finding roots of numbers.

Newton's method with $x_0 = 1$:



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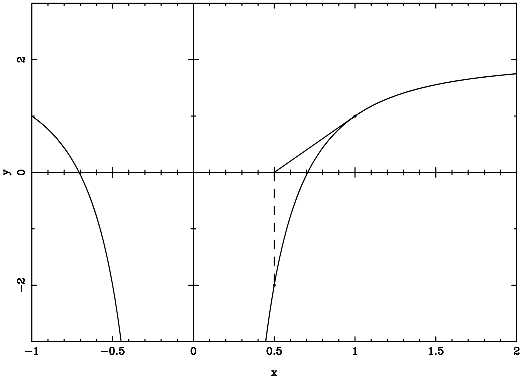

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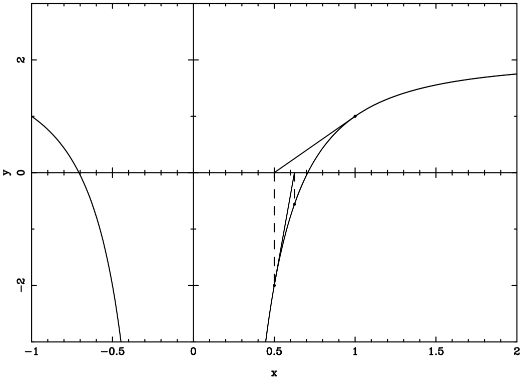

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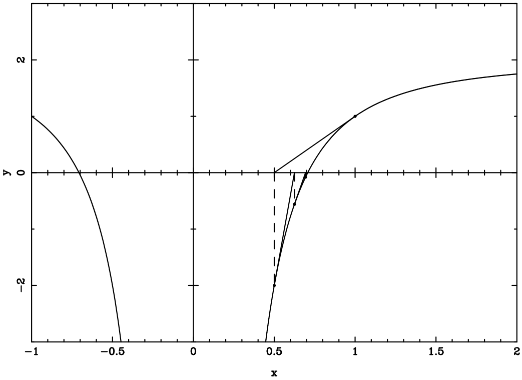

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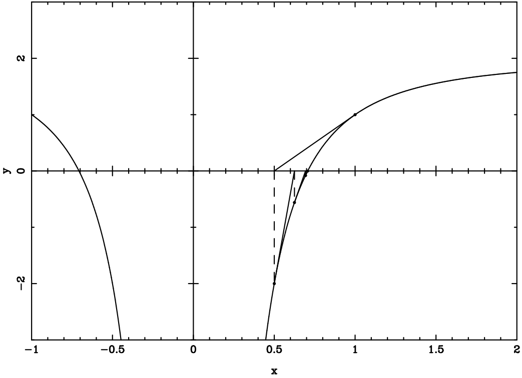

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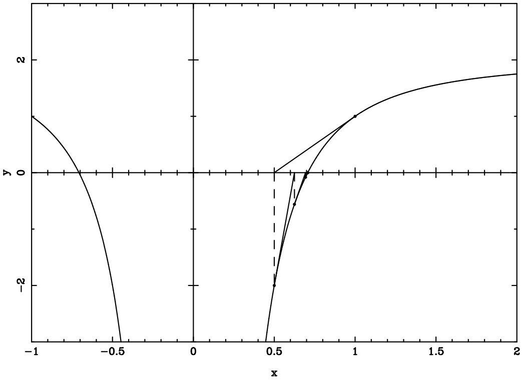
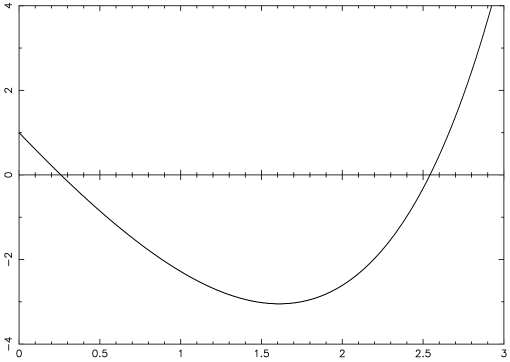


Convergence

$$
\begin{aligned}
x_0 &= 1.0 & \text{error} &= 0.292893231 \\
x_1 &= 0.5 & \text{error} &= 0.207106769 \\
x_2 &= 0.625 & \text{error} &= 0.0821067691 \\
x_3 &= 0.693359375 & \text{error} &= 0.0137473941 \\
x_4 &= 0.706708491 & \text{error} &= 0.0003982782 \\
x_5 &= 0.707106471 & \text{error} &= 0.0000002980
\end{aligned}
$$

The error is roughly squared each step. Very rapid convergence at end.

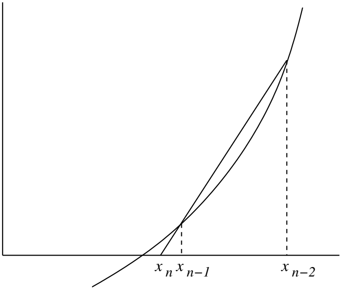
How would it work on our first example, finding the roots of $f(x) = e^x - 5x = 0$?



Works at both roots, but maybe a difficulty in between.

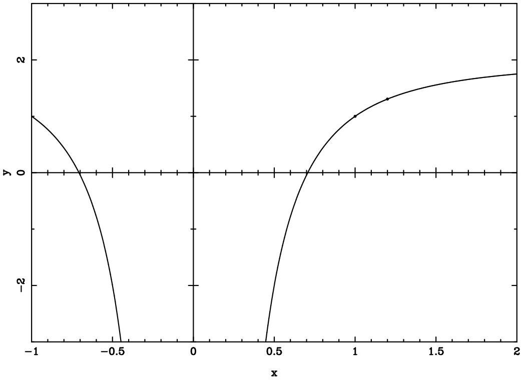
Secant Method

Approximate the gradient at x_{n-1} using $f(x)$ evaluated at x_{n-1} and x_{n-2}:



$$x_n = x_{n-1} - f\left(x_{n-1}\right)\left(\frac{f\left(x_{n-2}\right) - f\left(x_{n-1}\right)}{x_{n-2} - x_{n-1}}\right)^{-1}$$

Secant method with $x_0 = 1.2$ and $x_1 = 1.0$:



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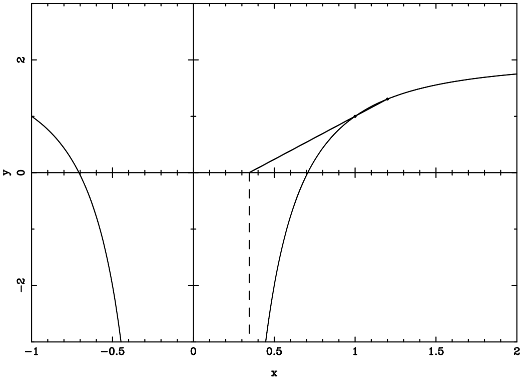

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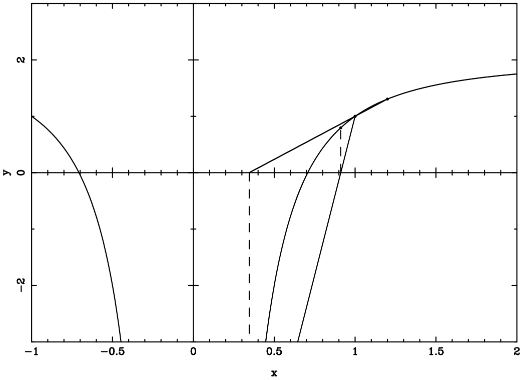

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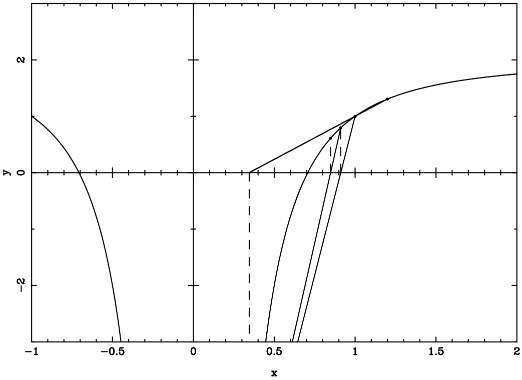

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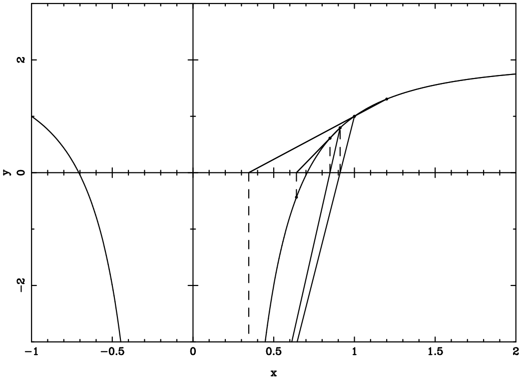

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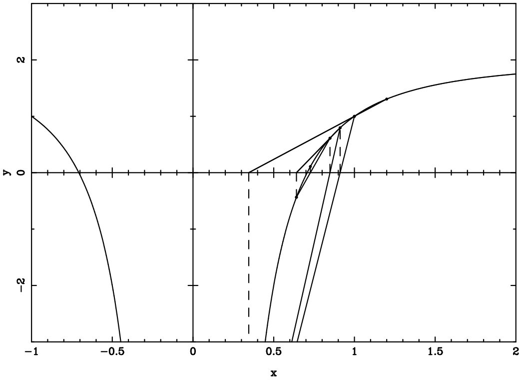

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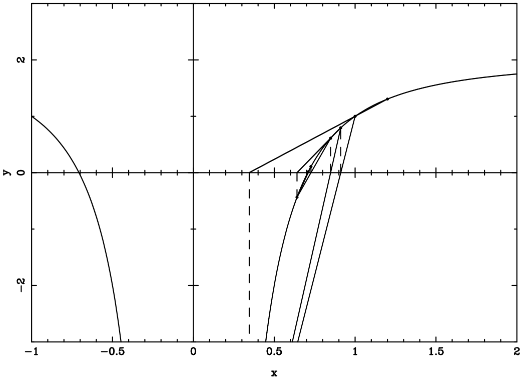

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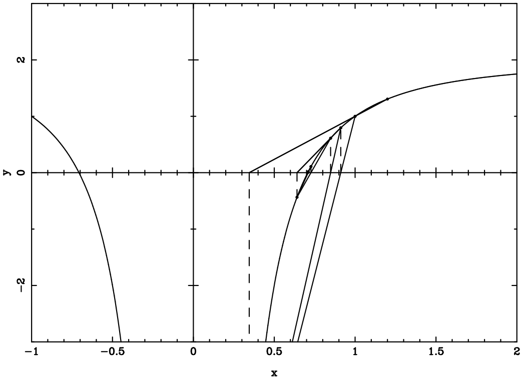

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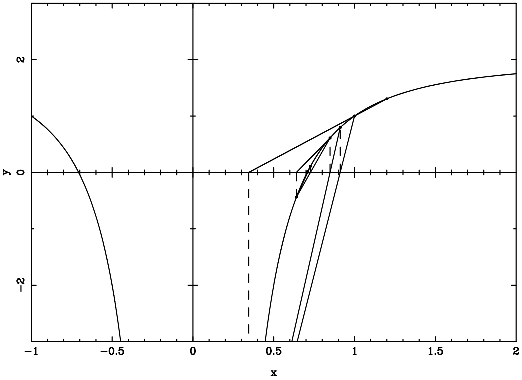

Convergence

$$
\begin{aligned}
x_0 &= 1.2 & \text{error} &= 0.492893279\\
x_1 &= 1.0 & \text{error} &= 0.292893231\\
x_2 &= 0.345454454 & \text{error} &= 0.361652315\\
x_3 &= 0.911302269 & \text{error} &= 0.204195499\\
x_4 &= 0.848540545 & \text{error} &= 0.141433775\\
x_5 &= 0.640884161 & \text{error} &= 0.066222608\\
&\vdots & &\vdots\\
x_9 &= 0.707107484 & \text{error} &= 0.000000715
\end{aligned}
$$

After initial messing about you get rapid convergence at the end.

Interval Halving

A more robust, but non-iterative technique for continuous $f(x)$:

1. Find a and b with $f(a)f(b) < 0$. Root is in the interval (a, b).
2. Evaluate $c = (a + b)/2$ and find $f(c)$. This is $f(x)$ at the mid-point.
3. If $f(c) = 0$ stop. You have found you root! (unlikely)
4. If $f(a)f(c) < 0$ the root is in interval (a, c). Replace b by c and go to step 2.
5. We must have $f(c)f(b) < 0$ so the root is in the interval (c, b). Repalce a by c and go to step 2.

Each time you go around the loop the interval where the root is located will be halved in length. Carry on as many time as you need to for desired accuracy.