Computational Mathematics/Information Technology

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2009–10

We have been looking at finding roots of equations of the form

$$f(x) = 0$$

However, we were originally interested in looking at systems of equations:
Set

$$\mathbf{x} = (x_1, x_2, \ldots, x_N) \quad \mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_N(\mathbf{x}))$$

Or more briefly:

$$\mathbf{f}(\mathbf{x}) = \mathbf{0}$$

where the zero vector $\mathbf{0} = (0, 0, \ldots, 0)$.

Some examples are relatively simple:

$$2x + y = 3, \quad x + 3y = 5.$$

Systems like this usually present very few problems (but not always).

Some are more difficult problems to solve, such as when the simultaneous equations are not linear:

$$f_1(x_1, x_2) = 2 - x_1^2 - x_2 = 0 \quad \text{and} \quad f_2(x_1, x_2) = 2x_1 - x_2^2 - 1 = 0.$$

In both of these cases we can think of each equation representing a line or curve in the x–y plane:

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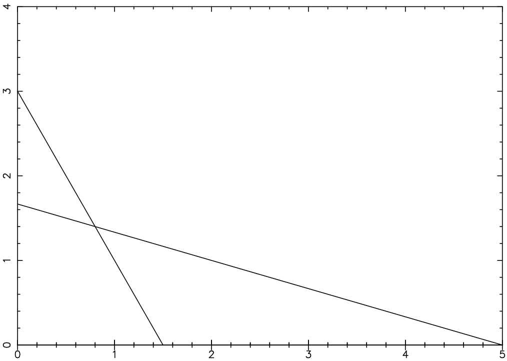

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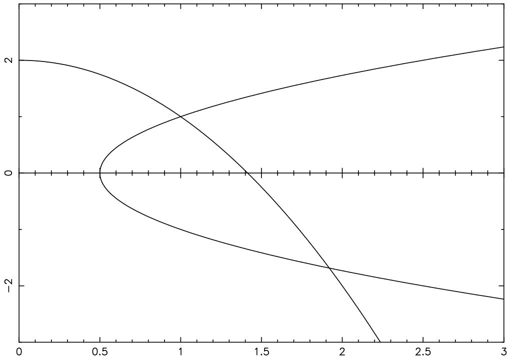

Previous we stated that we need to consider the following three questions.

- Does a solution exist? *existence*
- Is there one solution or many? *uniqueness*
- If using method of approximating the root how big is the error? *accuracy*

These are harder to answer when we have more than 1 variable and 1 equation!

Linear simultaneous equations

We will first look at the familiar problem of linear simultaneous equations. Consider the problem:

$$2x + y = 3$$

$$x + y = 2$$

Re-write in matrix form:

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Or as

$$A\mathbf{x} = \mathbf{b}$$

where

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

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Definitions: A is called a **square matrix** or a two by two matrix (denoted 2×2) since it has two rows and two columns. In general a matrix with m rows and n columns is called an m by n matrix (denoted $m \times n$)

$\mathbf{x}$ and $\mathbf{b}$ are referred to as two by one matrices (denoted 2×1) or **column matrices** or **column vectors**.

Consider the simple problem of solving $ax = b$ where a, b and x are real numbers. Formally

$$ax = b \quad \Rightarrow \quad a^{-1}ax = a^{-1}b \quad \Rightarrow \quad x = a^{-1}b = \frac{b}{a}$$

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Is it possible to do this with matrices?
Can we do

$$A\mathbf{x} = \mathbf{b} \quad \Rightarrow \quad A^{-1}A\mathbf{x} = A^{-1}\mathbf{b} \quad \Rightarrow \quad \mathbf{x} = A^{-1}\mathbf{b} \quad ?$$

We need to be able to do some matrix algebra...

Back to our example:

We have $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

The matrix corresponding to A^{-1} is given by

$$A^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

We can verify this by considering the following product:

$$A^{-1}A = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

The matrix we have denoted I — **the identity matrix** — corresponds to the number 1 when doing ordinary arithmetic.

The identity matrix has no effect when multiplying other matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

Note also

$$AA^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

so

$$A^{-1}A = AA^{-1} = I$$

So if we can find A^{-1} for a given A we are done...

Matrix inverses

For 2×2 matrices:

- For a general 2×2 matrix we can calculate the inverse as follows:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \Rightarrow \quad A^{-1} = \frac{1}{(ad - bc)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

  Clearly we require

$$ad - bc \neq 0.$$

- We can not calculate the inverse of matrix that isn't square.
- Not all square matrices have an inverse, as we can see above for the 2×2 case no inverse exists when $(ad - bc) = 0$. This last fact is analogous to not being able to calculate a^{-1} when $a = 0$.

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Finding the solution to

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is the same as finding the point of intersection of the lines

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The slopes of these two lines are

$$-a/b \qquad \text{and} \qquad -c/d$$

If these are the same then

$$ad - bc = 0$$

If

$$ad - bc = 0$$

the two lines are parallel.
Then either

1. The lines never meet — no solutions
2. The lines are the same — all points on the line are solutions. An infinite number of solutions.

In both cases you do not get a *unique solution*.

For larger square matrices:

$$A^{-1} = \frac{\text{Adjoint } A}{\text{Determinant } A}$$

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For 2×2 matrices it reduces to previous result.
For 3×3 matrices it is quite long and tedious to work out.
For $n \times n$ matrices with $n \geq 4$ it is definitely not the quickest way.
(Effort involved goes like $n!$, other methods go like n^3)

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Get the computer to do it!