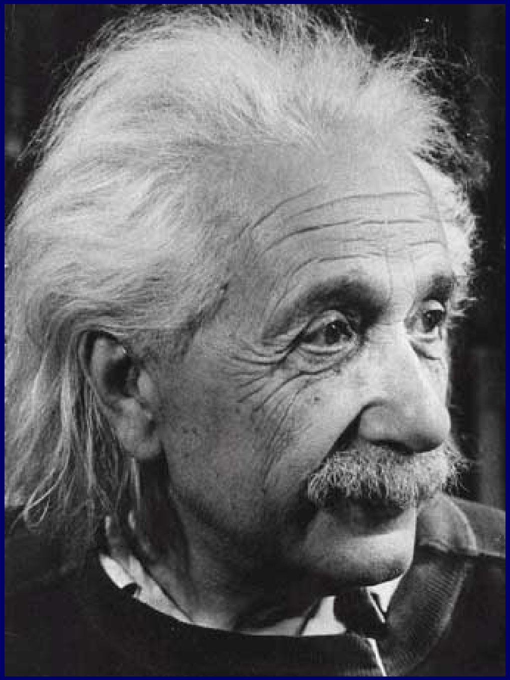
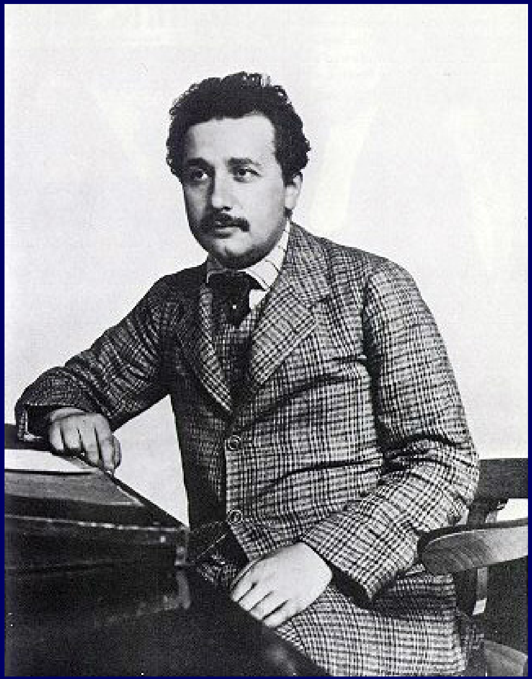
$E = mc^2$ made simple

Assuming you know
a little geometry and integration . . .



In 1905 Albert Einstein (1879–1955) published four papers on Brownian Motion, the Photoelectric Effect (for which he was awarded the Nobel Prize) and two on Special Relativity which included the equation $E = mc^2$.



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Outline

- Newton's world
- The problem

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- Width does not change
- Clocks go slower
- Things get shorter
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- Things get heavier
- And finally ... $E = mc^2$

Newton's world

Very much as we experience the world and see things:

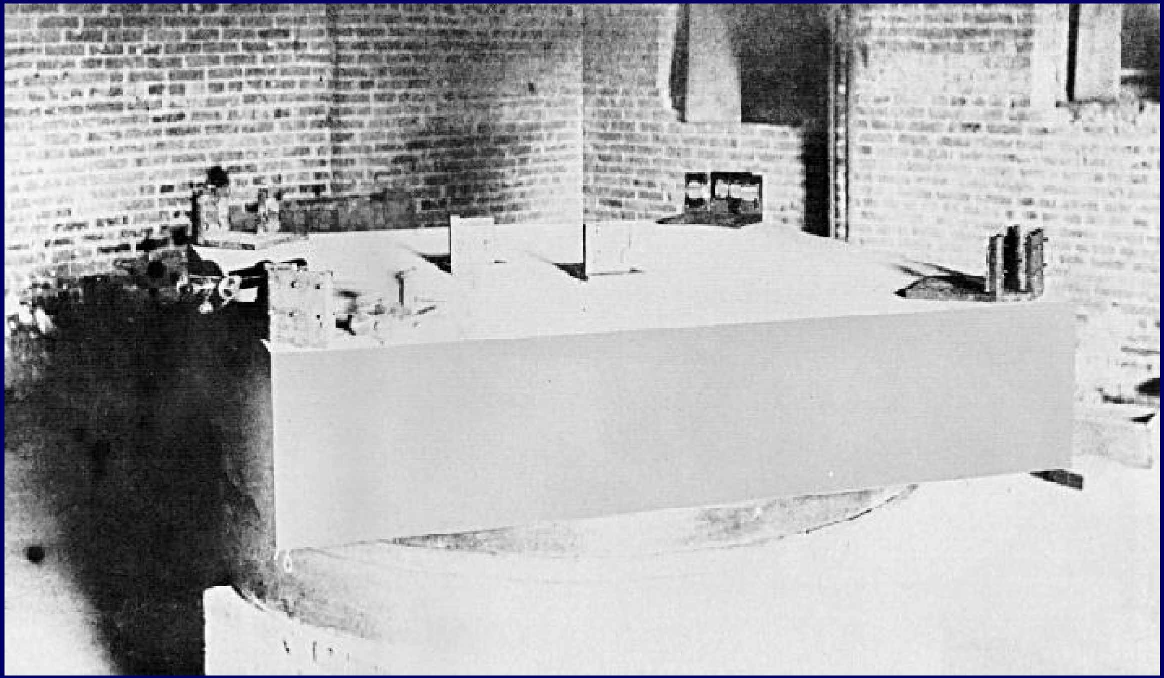
- Universal time — all clocks run at the same rate.
- Universal reference frame — some stationary coordinate system in which everything moves.

Newton's laws

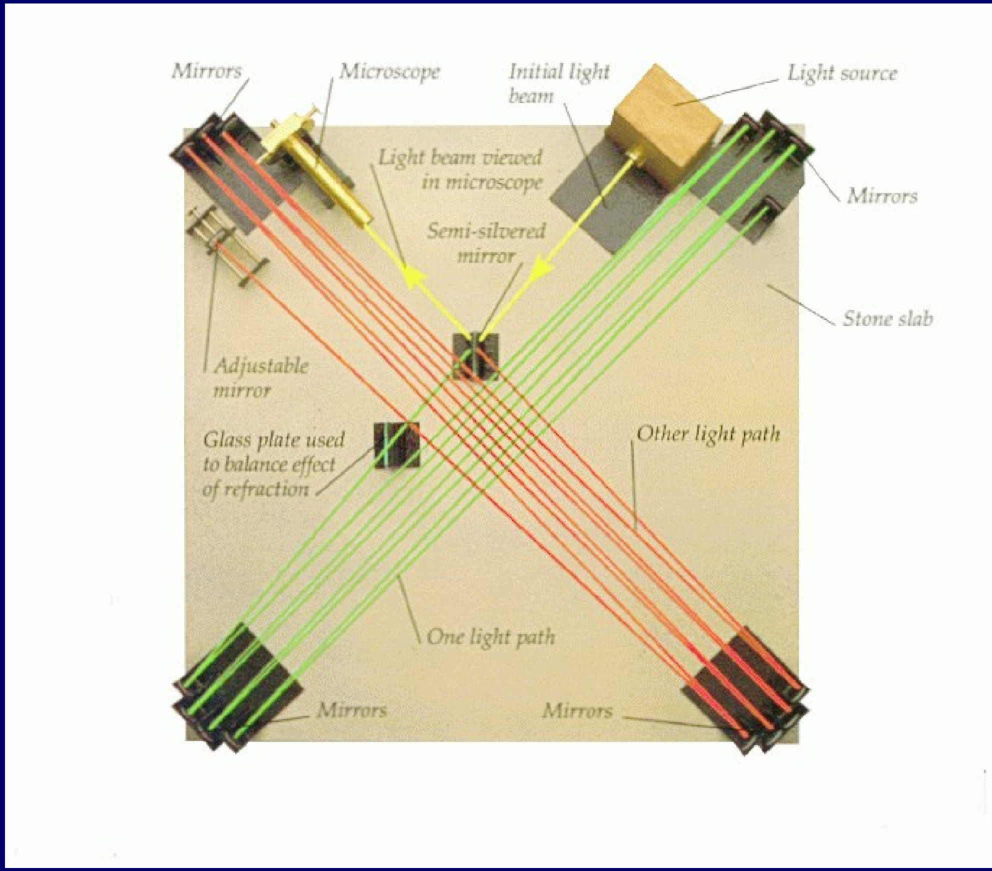
- All objects move in a straight line with constant speed unless acted on by a force.
- The rate of change of momentum (mass$\times$velocity) of an object (or system) is proportional to the force applied.
- For every action there is an equal and opposite reaction.

All of these hold true and are not contradicted by Einstein's special relativity.

The problem — Michelson-Morley



An attemp to measure changes in the speed of light depending on the direction we are moving.



No matter how the observer was moving the speed of light was always a constant c in all directions.

The solution — Relativity

- There is no universal stationary frame of reference.
- Any observer moving with constant velocity will see the same laws of physics operating in their own frame of reference as any other observer. Everything happens relative to the observer.

  An experimenter in a sealed spaceship drifting in space would be unable to determine how fast and in what direction the spaceship was moving.

Width remains constant

Take some balls and a set of holes that are only just bigger than the holes.

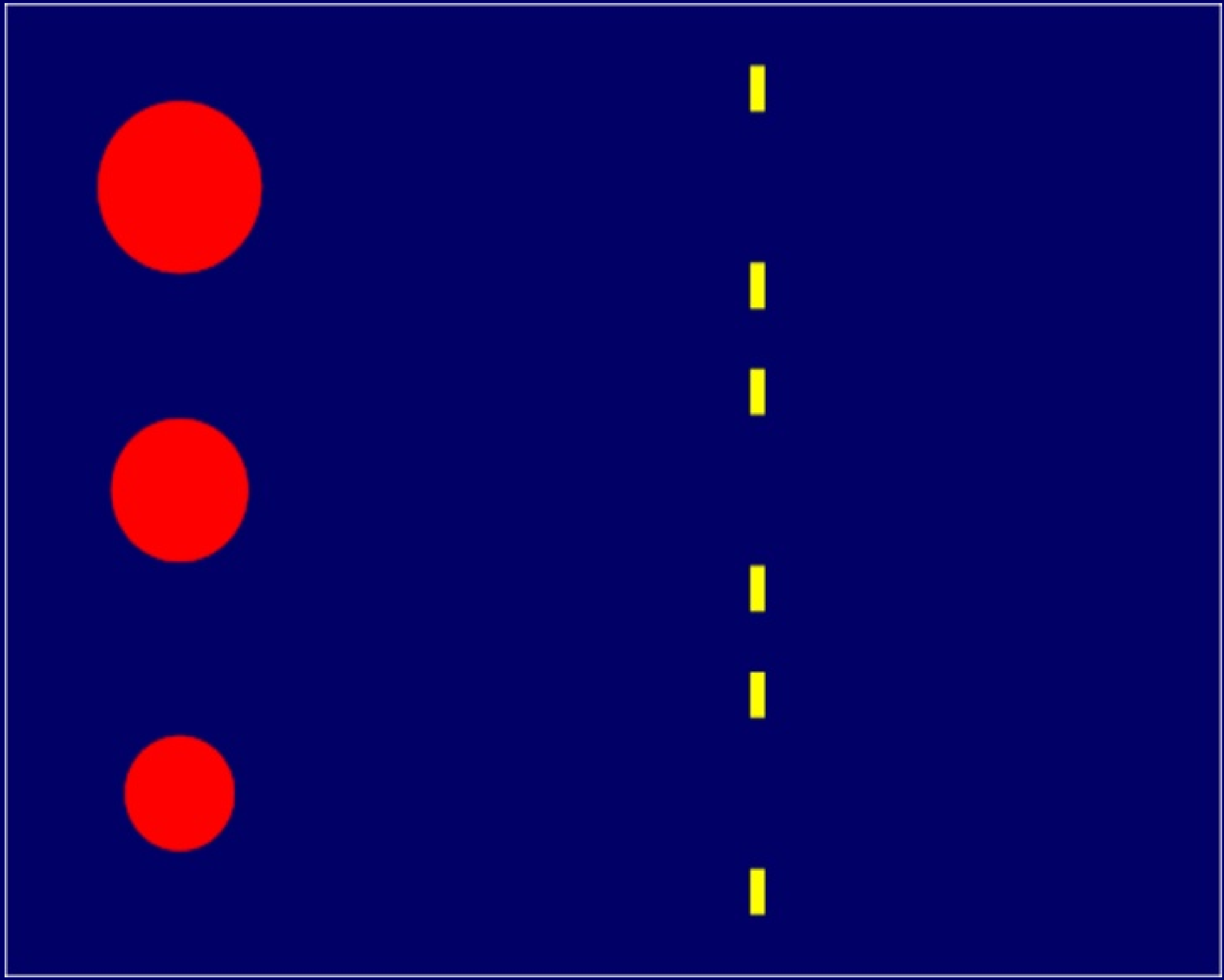
What happens if you fire the balls at the holes fast?

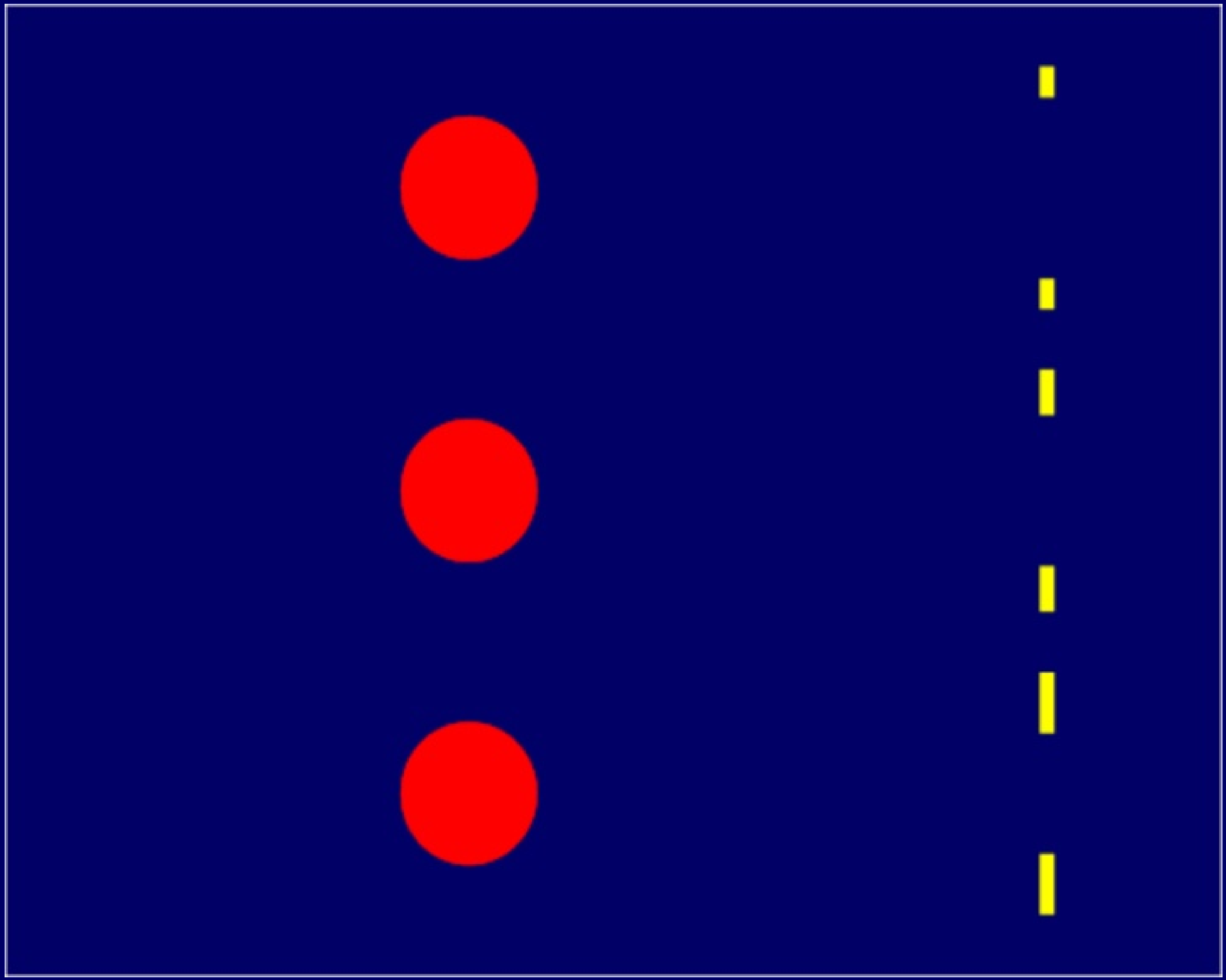
Width remains constant

Take some balls and a set of holes that are only just bigger than the holes.

What happens if you fire the balls at the holes fast?

They could get wider, not change width, or get narrower . . .



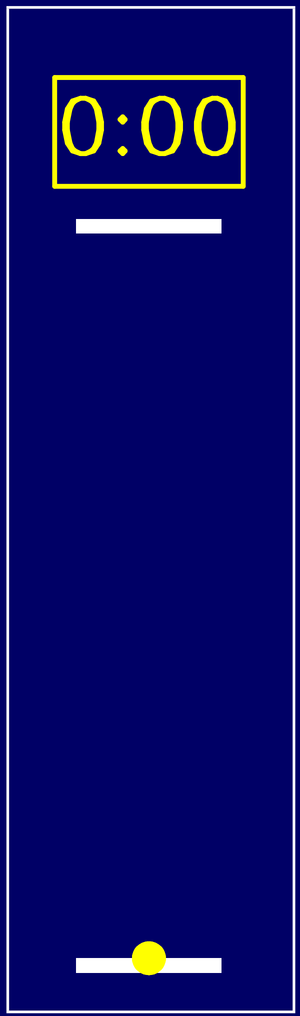


Moving objects do not change width.

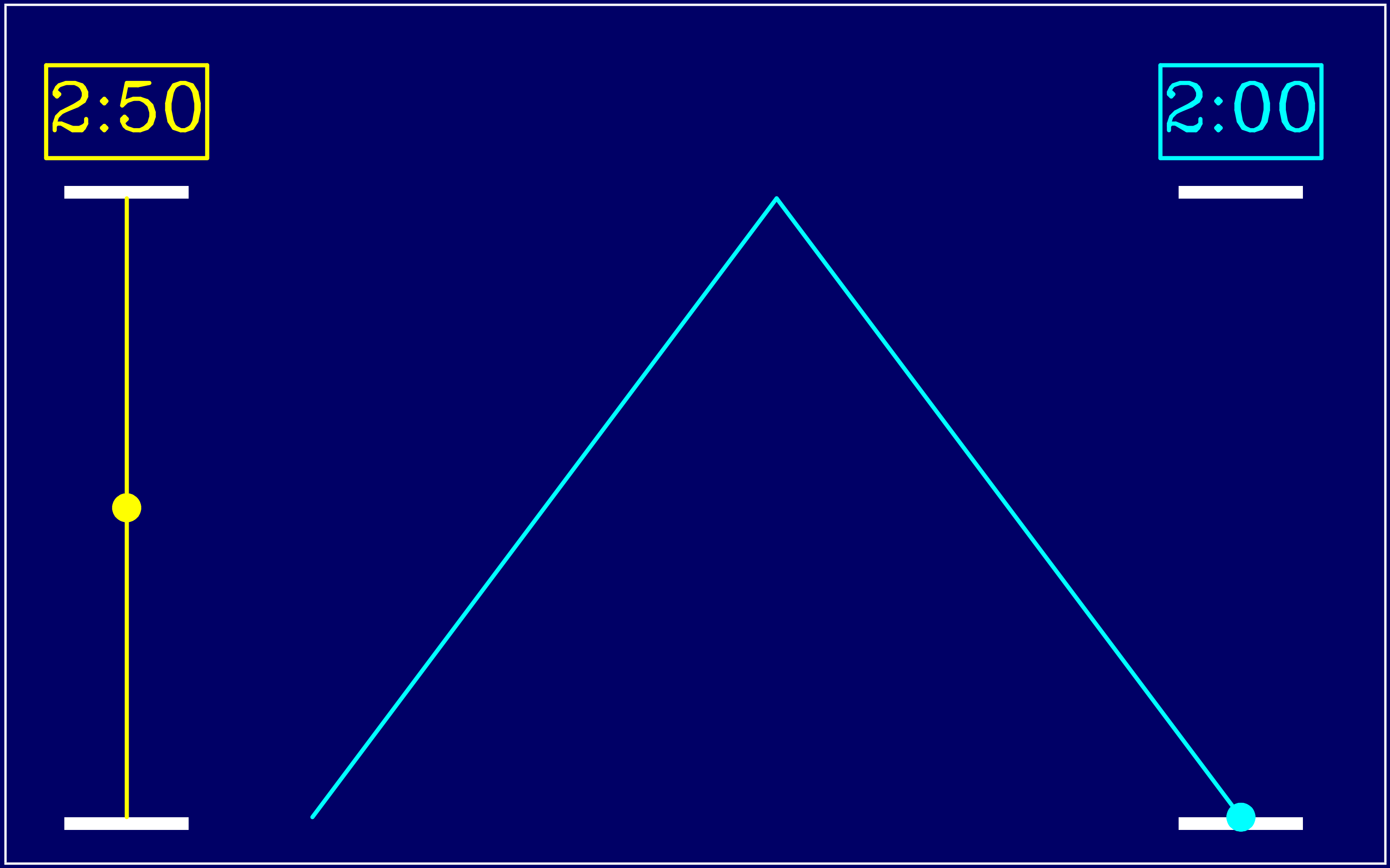
Clocks go slower

All frames of reference are equivalent. The speed of light measured by any observer will be the same — c.

Imagine that you made clocks that measured time by counting the reflections of a pulse of light between two mirrors. All observers moving with such clocks would think that they ran at the same rate.



Two identical light clocks, one stationary and one moving to the right with speed v.



A stationary observer will see the moving clock running more slowly than the stationary clock.

If the stationary clock is of length L then the time taken for the light to travel from one mirror to the other is

$$t = L/c.$$

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The stationary observer sees the light pulse in the moving clock take a longer path and so takes a longer time, t', to travel between the mirrors. The mirrors will have moved sideways a distance vt'. By Pythagoras, the length of the diagonal will be

$$\sqrt{L^2 + (vt')^2}.$$

The time taken for the light to travel down this path is

$$t' = \frac{\sqrt{L^2 + (vt')^2}}{c}.$$

A bit of rearranging gives

$$t' = \frac{t}{\sqrt{1 - (v/c)^2}}.$$

Moving clocks run more slowly.

Things get shorter

Now consider what would happen if there were two light clocks — one perpendicular to the direction of travel and one aligned with it . . .

An observer moving with the two clocks will see them as running at the same rate. Pulses of light would remain sychronised.

Things get shorter

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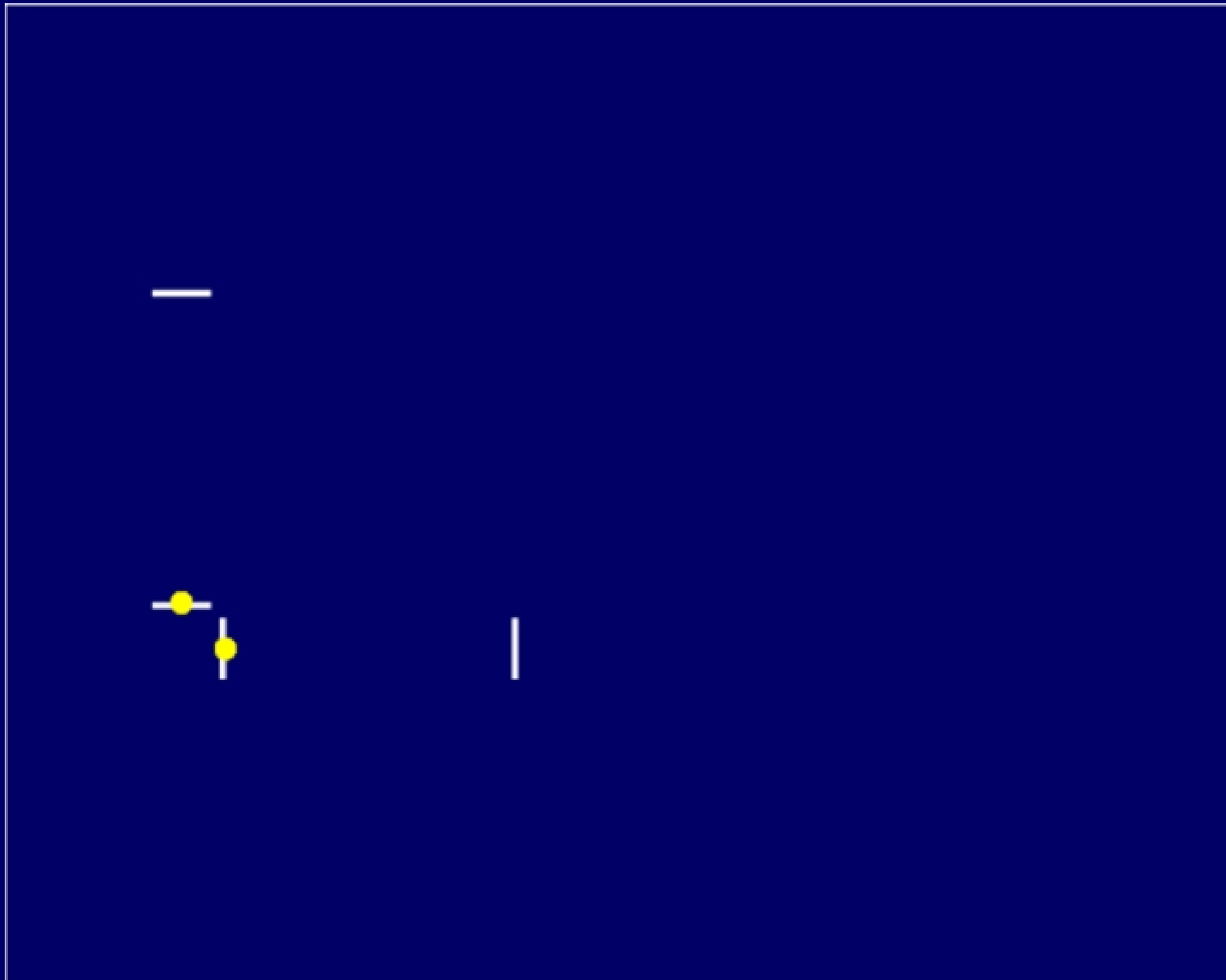
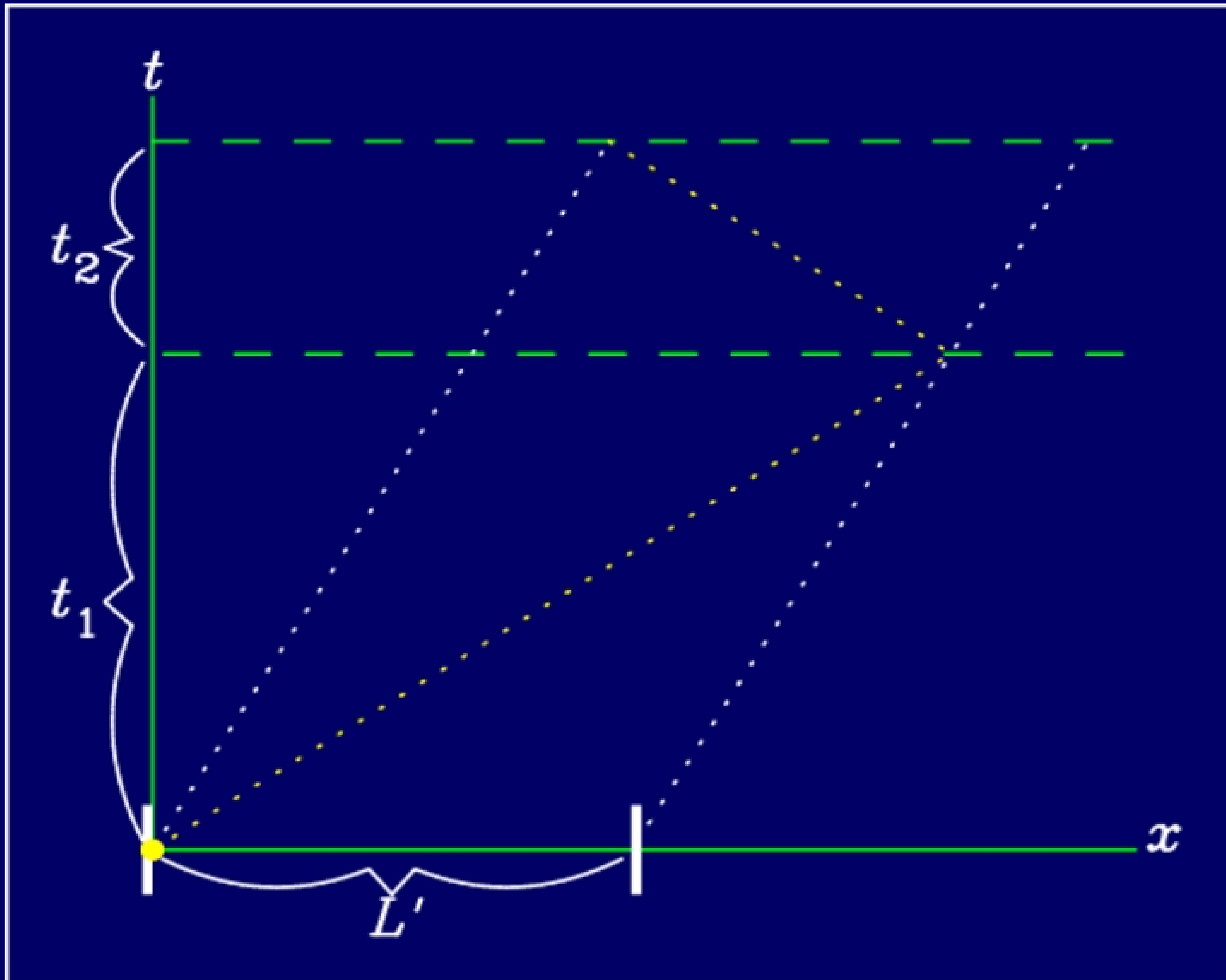
An observer moving with the two clocks will see them as running at the same rate. Pulses of light would remain sychronised.

A stationary observer would also see the clocks staying sychronised. But if the clocks had the same length . . .



The only way stationary observer could see the two clocks running at the same rate is if the one pointing in the direction of travel changed length.

Assume this clock as seen by the stationary observer is now of length L'.



t
t2
t1
x
L'

The time for the pulse to travel to the right is t_1.
The time for the pulse to travel to the left is t_2.

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The right traveling pulse will travel a distance

$$L' + vt_1 = ct_1,$$

while the left traveling pulse will travel a distance

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$$L' - vt_2 = ct_2.$$

From these we find

$$t_1 = \frac{L'}{(c-v)} \quad \text{and} \quad t_2 = \frac{L}{(c+v)}.$$

The total time is

$$t_1 + t_2 = \frac{2cL'}{(c^2 - v^2)} = \frac{2L'}{c(1 - (v/c)^2)}.$$

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Earlier we saw the time taken for the moving perpendicular clock to register two reflections is

$$2t' = \frac{2t}{\sqrt{1 - (v/c)^2}} = \frac{2L}{c\sqrt{1 - (v/c)^2}}.$$

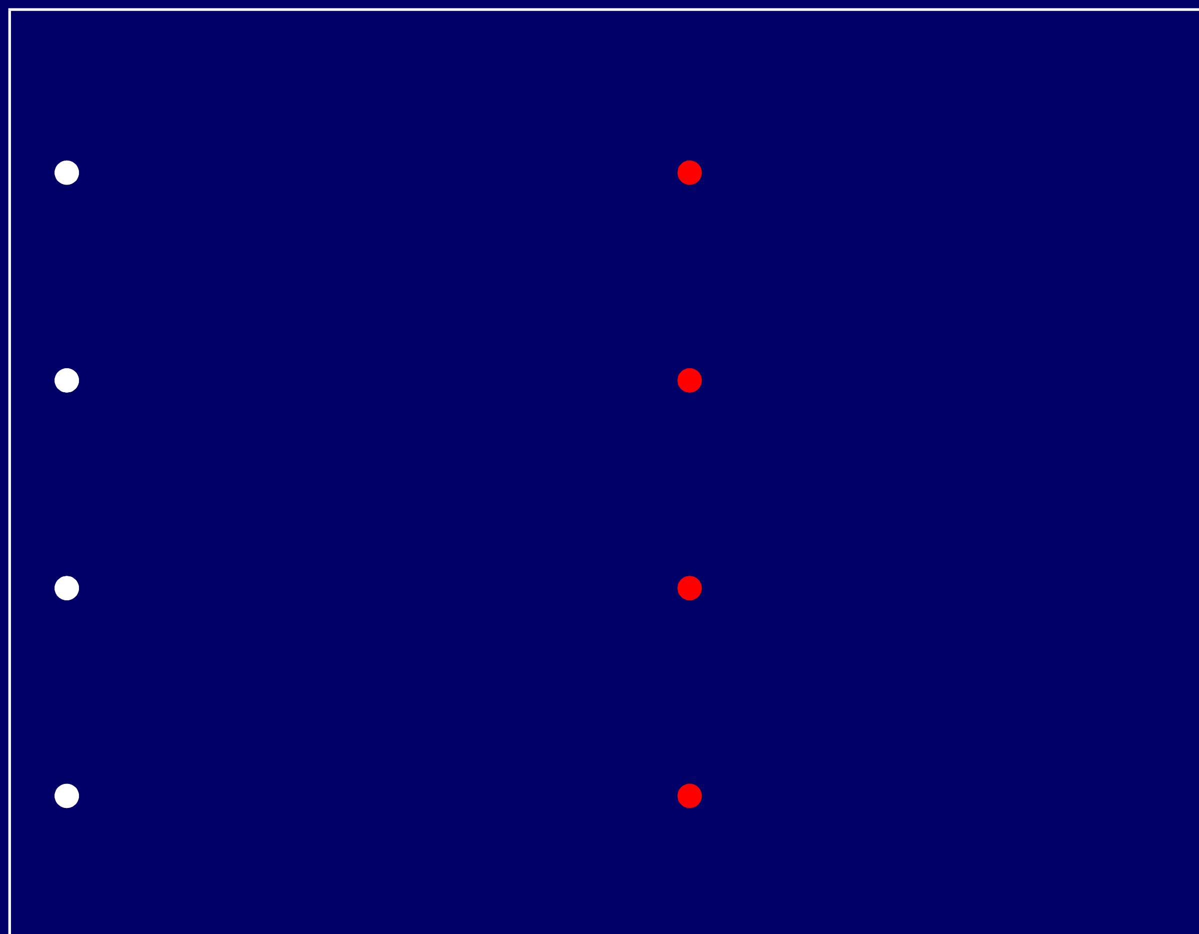
This and the expression for $t_1 + t_2$ must be the same, so

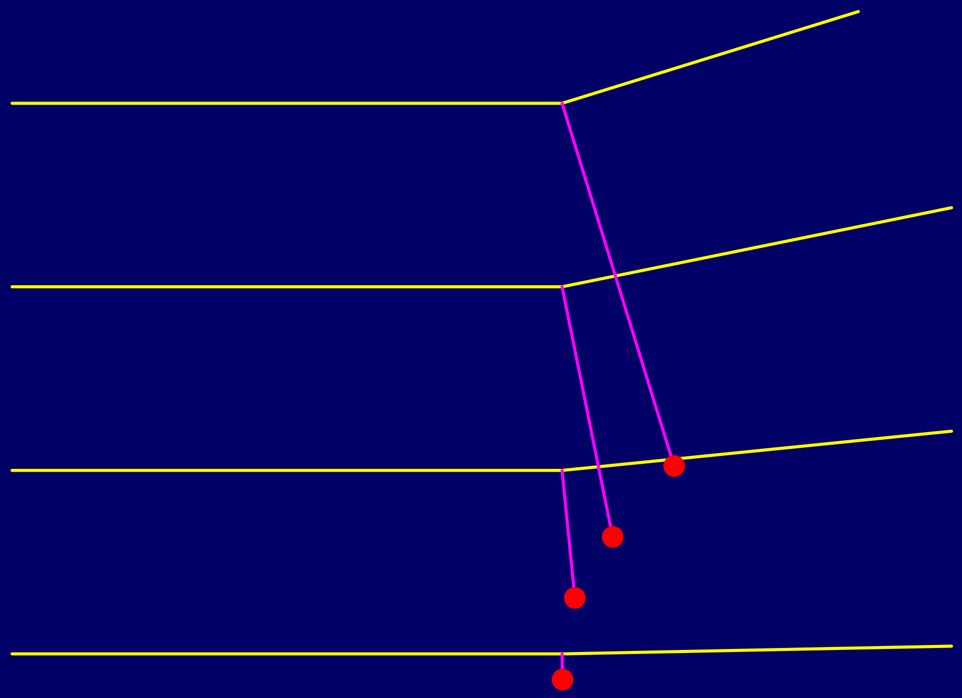
$$L' = L\sqrt{1 - (v/c)^2}.$$

Moving objects get shorter.

Things get heavier

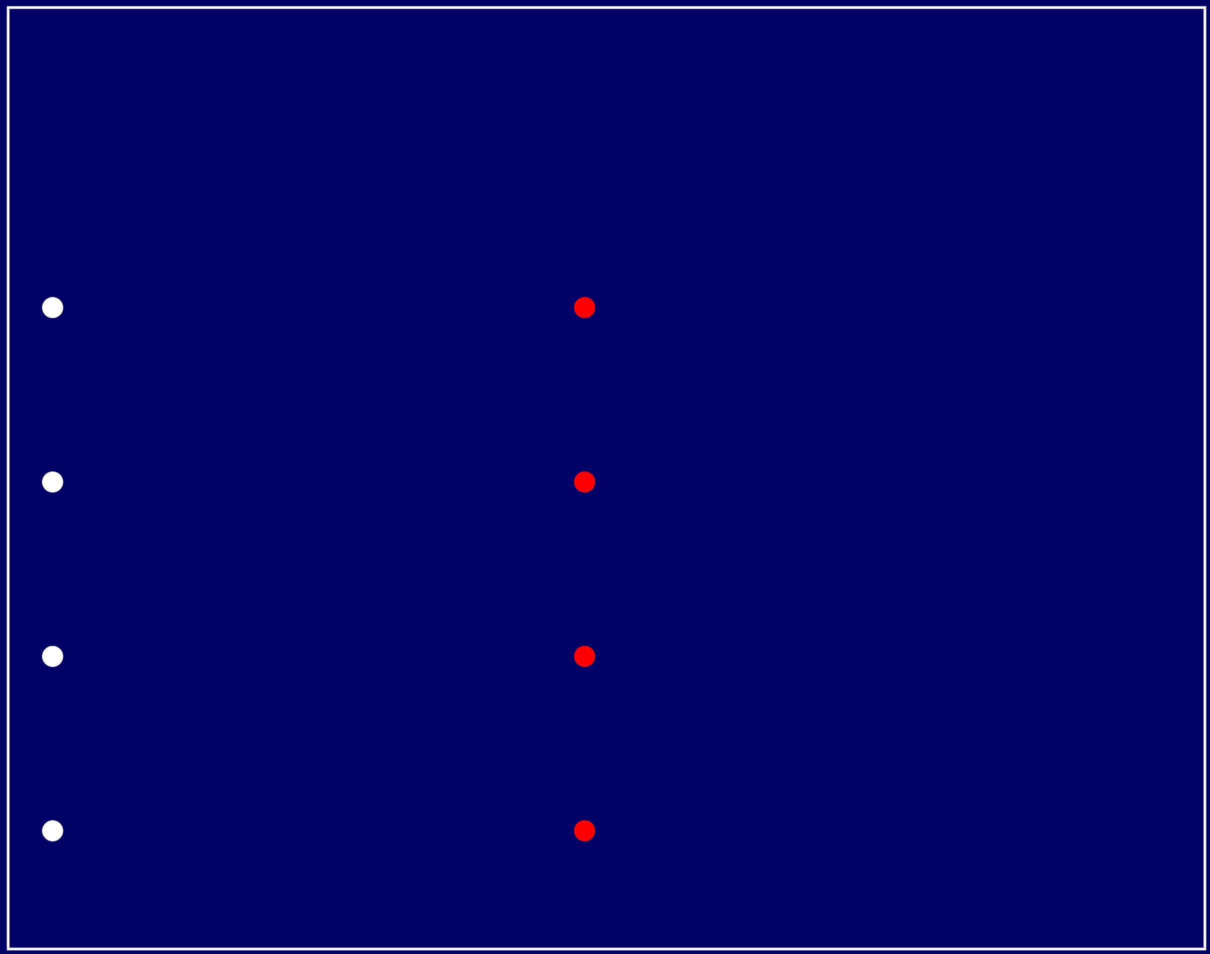
First some Newtonian billiards —

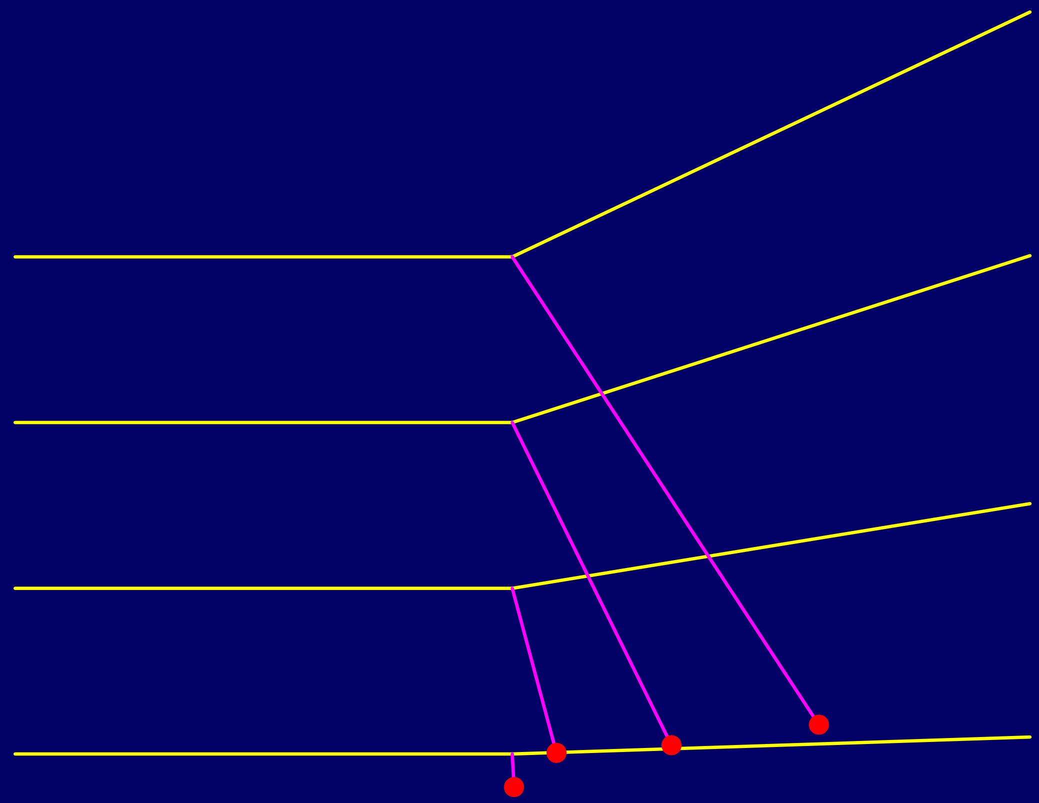




- Trajectories after a collision are perpendicular.
- In the weakest collisions the trajectory of the hit ball is nearly perpendicular to that of the incoming ball.

Relativistic billiards with the incoming "balls" having speed $v = 0.8 \times c$.





- Trajectories not perpendicular after the collision.
- As the collisions get weaker the trajectory of the hit ball *still* tends to being perpendicular to the incoming ball.

Imagine the identical balls collide, but only just...

- The ball that was originally stationary in the stationary frame will move slowly in a direction effectively perpendicular to the incoming path of the other ball. Say it moves down with speed u.

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- The other will be slightly deflected and vertical component of its velocity will be u' as observed by the stationary observer.
- But an observer moving with the incoming ball will see things in reverse: his ball will move slowly up with a speed u after the collision.

- The time taken for the moving observer to see this ball move a perpendicular distance l will be l/u.

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- The stationary observer will see it take the time l/u'.
- But the stationary observer sees things happening more slowly in the moving frame, so

$$\frac{l}{u} = \frac{l}{u'} \times \sqrt{1 - (v/c)^2},$$

or

$$u' = u\sqrt{1 - (v/c)^2}.$$

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Back to Newton's laws — conservation of momentum says that the total momentum in the transverse direction is unchanged by the collision. We have

$$0 = m'u' - mu,$$

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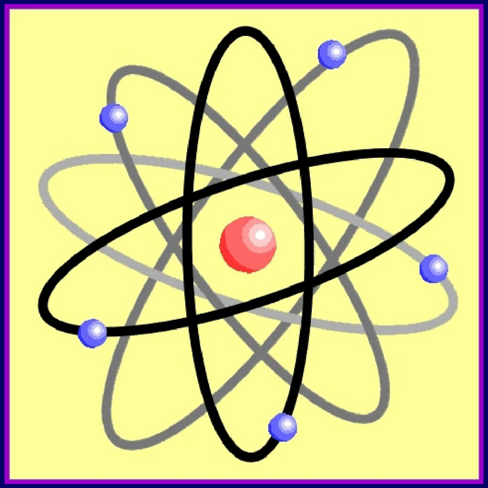
$$m' = \frac{m}{\sqrt{1-(v/c)^2}}.$$

Moving objects get heavier.

Jumbo Jet

GB Coxless 4 at the Olympics

Jumbo Jet



Atom

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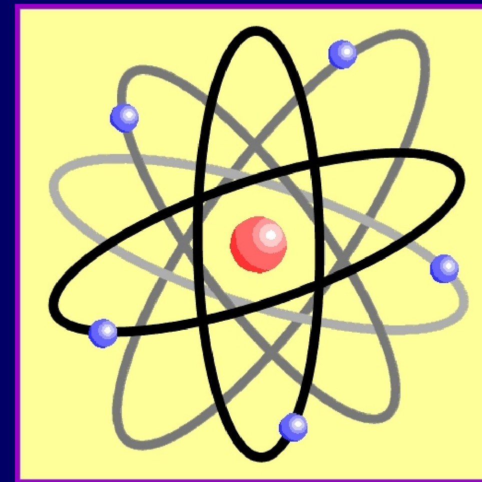
Jumbo Jet



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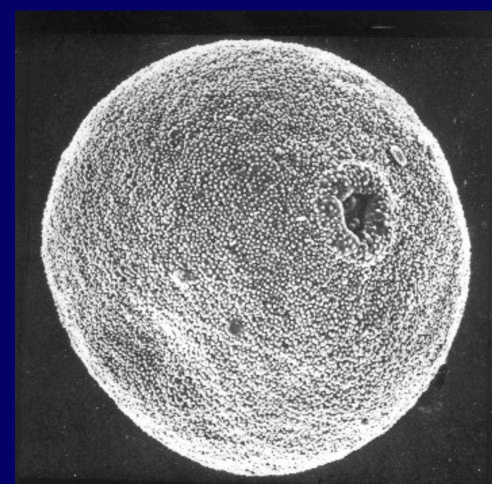

Grass Pollen

And finally...

$$\text{Change in Energy} = \text{Work Done} = \text{Force} \times \text{Distance}$$

And finally...

Change in Energy $=$ Work Done $=$ Force $\times$ Distance

In these calculations we will apply a force to an object that is initially at rest. After a time T, it will have moved a distance X and have attained a speed V. Initially its mass will be m_0 and at the end $m_V = m_0 / \left(1 - (V/c)^2\right)^{1/2}$.

Its energy at speed v will be $E(v)$.

$$E(V) - E(0) \ = \ \int_0^X F\,dx = \int_0^X \frac{d(mv)}{dt}\,dx$$

$$
\begin{aligned}
E(V) - E(0) &= \int_0^X F\,dx = \int_0^X \frac{d(mv)}{dt}\,dx \\
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&= m_V V^2 - \int_0^V \frac{m_0 v}{\left(1-(v/c)^2\right)^{1/2}}\, dv \\
&= m_V V^2 + \left[m_0 c^2 \left(1-(v/c)^2\right)^{1/2} \right]_0^V
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&= m_V V^2 + m_0 c^2 \left(1-(V/c)^2\right)^{1/2} - m_0 c^2
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\end{aligned}
$$

This gives

$$E(V) - E(0) = \left(m_{_V} - m_{_0}\right) c^2$$

or

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The last bit is Einstein's "guess" that this relation should apply to the total mass and not just the change in mass, giving . . .

$$E = mc^2$$