Mathematical Methods: Fourier Series

Fourier Series: The Basics

Fourier series are a method of representing periodic functions. It is a very useful and powerful tool in many situations. It is sufficiently useful that when some non-periodic problems arise transformations are used to make such problems periodic so the Fourier series can be used.

A function with period P means that $f(x) = f(x+P)$ for all x. Some well known periodic functions are listed below:

Function	Period	Function	Period

Note: If a function is periodic with period P then it also has periods $2P$, $3P$, $4P$, and so on. If we set $2\pi/a = P$, or $a = 2\pi/P$, we see from the above table that $\cos(2\pi x/P)$ and $\sin(2\pi x/P)$ both have period P. Similarly, $\cos(2n\pi x/P)$ and $\sin(2n\pi x/P)$, $n = 1, 2, 3, \ldots$, both have period P/n and hence also have period P.

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$\cos x$	2π		
$\sec x$	2π		
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The essential idea behind Fourier series is to represent periodic functions in terms of a sum of well known periodic functions. Sines and cosines are chosen as they are smooth. If $f(x)$ has period P we write

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{2n\pi x}{P} + b_n \sin \frac{2n\pi x}{P}.$$

Some people use a convention where in stead of a_0 they have $a_0/2$. Always check to see which convetion an author is using.

This leaves the problem of how to find a_0, a_n and b_n for a given function ...

To find a_0 we integrate $f(x)$ over any interval of length P, say from 0 to P:

$$\int_0^P f(x)\,dx = \int_0^P \left(a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{P} + b_n \sin\frac{2n\pi x}{P} \right)\,dx$$

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Assuming you can swap the order of integration and the summations we get

$$\int_0^P f(x)\,dx = \int_0^P a_0\,dx$$

$$+ \sum_{n=1}^{\infty} \left(a_n \int_0^P \cos\frac{2n\pi x}{P}\,dx + b_n \int_0^P \sin\frac{2n\pi x}{P}\,dx \right).$$

But

$$\int_0^P \cos\frac{2n\pi x}{P}\,dx = \left[\frac{P}{2n\pi}\sin\frac{2n\pi x}{P}\right]_0^P = \frac{P}{2n\pi}\left(\sin 2n\pi - \sin 0\right) = 0,$$

and

$$\int_0^P \sin\frac{2n\pi x}{P}\,dx = \left[-\frac{P}{2n\pi}\cos\frac{2n\pi x}{P}\right]_0^P = -\frac{P}{2n\pi}\left(\cos 2n\pi - \cos 0\right) = 0,$$

giving

$$\int_0^P f(x)\,dx = Pa_0 \qquad \text{or} \qquad a_0 = \frac{1}{P}\int_0^P f(x)\,dx. \tag{1}$$

So a_0 is just the average value of $f(x)$.

To find all the other a_n and b_n we need to use the trigonometrical relations

$$\cos A \cos B = \frac{1}{2}\left(\cos(A+B) + \cos(A-B)\right),$$

$$\sin A \sin B = \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right),$$
$$\sin A \cos B = \frac{1}{2}\left(\sin(A+B) + \sin(A-B)\right).$$

To find a_n consider

$$\int_0^P \cos\frac{2m\pi x}{P} f(x)\,dx$$

$$= \int_0^P \cos\frac{2m\pi x}{P}\left(a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{P} + b_n \sin\frac{2n\pi x}{P}\right)\,dx.$$

Again, assuming we can swap the order of integration and summation, we obtain

$$\int_0^P \cos\frac{2m\pi x}{P} f(x)\,dx = a_0 \int_0^P \cos\frac{2m\pi x}{P}\,dx$$

$$+\sum_{n=1}^{\infty} a_n \int_0^P \cos\frac{2m\pi x}{P}\cos\frac{2n\pi x}{P}\,dx + b_n \int_0^P \cos\frac{2m\pi x}{P}\sin\frac{2n\pi x}{P}\,dx,$$

$$
\begin{aligned}
&= a_0 \int_0^P \cos \frac{2m\pi x}{P}\, dx \\
&+ \sum_{n=1}^{\infty} \frac{a_n}{2} \int_0^P \cos \frac{2(m+n)\pi x}{P} + \cos \frac{2(m-n)\pi x}{P}\, dx \\
&+ b_n \int_0^P \sin \frac{2(m+n)\pi x}{P} - \sin \frac{2(m-n)\pi x}{P}\, dx.
\end{aligned}
$$

Since m and n are both positive integers we have seen already that all these integrals are zero *except* for the cases where $m = n$. In the case $m = n$ the sine integral is obviously 0, but $\cos(2(m-n)\pi x/P) = 1$ and so that integral gives P. Hence

$$
\int_0^P \cos \frac{2m\pi x}{P} f(x)\, dx = \frac{a_m P}{2},
$$

or

$$a_n = \frac{2}{P}\int_0^P \cos\frac{2n\pi x}{P} f(x)\,dx. \tag{2}$$

Similarly we find

$$b_n = \frac{2}{P}\int_0^P \sin\frac{2n\pi x}{P} f(x)\,dx. \tag{3}$$

The integrals for a_0, a_n and b_n given above are all over the interval from 0 to P. However as all the functions involved are periodic with period P they can be taken over *any* interval of length P. You are free to choose the interval to make the calculations involved easier for youself.

Equations (1), (2) and (3) are called the Euler formulas for the Fourier coefficients.

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Example: Sketch the periodic function with period 2π given by

$$f(x) = \begin{cases} -1 & -\pi < x \leq 0 \\ +1 & 0 < x \leq \pi \end{cases}$$

Find its Fourier series.