Act2/SSwMS2/X2 Discrete Stochastic Modelling, June 2000: Solutions

Question 1

(i) A SRW is a process $\{X_n : n \geq 0\}$ given by $X_n = X_{n-1} + J_n$, where $\{J_n : n \geq 0\}$ is a sequence of i.i.d random variables taking values $+1$ (with prob. p) or -1 (with prob. $-p$).
A SSRW is a SRW for which $+1$ and -1 are equally likely.
The number of up-jumps made by X by time n has $\text{Bin}(n, p)$ distribution.
The independence of the J_n implies that the jump following an up-jump has probability p of being another up-jump or $1-p$ of being a down-jump, so the required distribution is $\text{Bin}(k, p)$.

(ii)

a) No. This would contradict the independence of the J_n.

b) The trader's transition matrix would be $P = \begin{pmatrix} .57 & .43 \\ .45 & .55 \end{pmatrix}$. The corresponding matrix for the SRW would be $\begin{pmatrix} p & 1-p \\ p & 1-p \end{pmatrix}$.
The SRW is much too simplistic to be reasonable. The 2-state Markov chain model is likely to be better, but it is by no means a complex model.

(iii)

a) Of 36 visits to state "+", 20 were followed by another such visit, so $\hat{p}_{++} = \frac{20}{36}$. Similarly $\hat{p}_{-+} = \frac{16}{43}$.
If the trader is right, the number of ++ sequences should be $\text{Bin}(36, 0.57)$, or approximately $N(20.52, 8.82)$. The observed value of 20 is clearly fine.
Similarly, the number of -+ sequences should be $\text{Bin}(43, 0.45)$, or approximately $N(19.35, 10.6)$. The observed value of 16 is roughly 1 s.d. from the mean, not enough to reject any hypothesis.

b) Not entirely. The question here is: given the middle value in a sequence of 3, is the last value affected by the first value? For example, with a + in the middle, the ratio of + to - in last place is 12 : 7 if a + comes first, or 7 : 9 if a - comes before. With a - in the middle the difference is not so noticeable—the ratio is 5 : 11 or 11 : 16.

c) State space { RTRY, RTFY, FTRY, FTFY }. Diagram.

$$\hat{P} = \begin{pmatrix} \frac{12}{19} & 0 & \frac{7}{19} & 0 \\ \frac{7}{16} & 0 & \frac{9}{16} & 0 \\ 0 & \frac{5}{16} & 0 & \frac{11}{16} \\ 0 & \frac{11}{27} & 0 & \frac{16}{27} \end{pmatrix}.$$

Question 2

(i)

a) A state i is periodic if, given that $X_0 = i$, the only times when X can again be at i are times which are multiples of an integer $d > 1$. We call X periodic if (some of) its states are periodic.

b) The chain not irreducible if there are two states i and j such that, starting from $X_0 = i$, it is impossible for X to reach j.

c) X is ergodic if it is aperiodic and irreducible.
Possible transition matrices (though there are many other possibilities) are:

$$a) \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix}; \quad b) \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad c) \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

(ii) a) The specification of the situation indicates a lack of dependence of future transitions on past states, given the present activity, implying that a Markov model is appropriate. The transition matrix is

$$P = \begin{pmatrix} 0.25 & 0.25 & 0.25 & 0.25 \\ 0 & 0.5 & 0.2 & 0.3 \\ 0 & 0 & 0.5 & 0.5 \\ 0 & 0.2 & 0.3 & 0.5 \end{pmatrix}.$$

b) All states are aperiodic, since $p_{ii} > 0$ for each i. However, the state space is not irreducible: state I is transient. Therefore X is not ergodic.

c) We need to solve $\pi^T P = \pi^T$ (though we can equally well use a 3×3 submatrix of P excluding the transient state I). This boils down to

$$\begin{array}{lllll} 0.25\pi_I & & & & = \pi_I \\ 0.25\pi_I & +0.5\pi_R & & +0.2\pi_T & = \pi_R \\ 0.25\pi_I & +0.2\pi_R & +0.5\pi_A & +0.3\pi_T & = \pi_A \\ 0.25\pi_I & +0.3\pi_R & +0.5\pi_A & +0.5\pi_T & = \pi_T \end{array}$$

The first equation implies that $\pi_I = 0$ (which was obvious anyway), the second that $\pi_T = 2.5\pi_R$, the third that $\pi_A = 1.9\pi_R$, and the fourth is redundant. Applying the condition that $\sum \pi_i = 1$ we have $1 = \pi_R(1 + 2.5 + 1.9)$, so that $\pi_R = \frac{10}{54}$, $\pi_A = \frac{19}{54}$ and $\pi_T = \frac{25}{54}$.

d) We have

$$PV = \begin{pmatrix} 1 & 2 & 0.25 & .5 \\ 1 & 2.8 & 0 & -.1 \\ 1 & -2 & 0 & -1 \\ 1 & 0.4 & 0 & .8 \end{pmatrix}.$$

We see that each column of PV is a multiple of the corresponding column of V, the factor being given by the corresponding diagonal entry in Λ.

$\mathbb{P}(X_n = R \,|\, X_0 = I)$ is the (I, R) entry of the matrix P^n, which may be calculated as $P^n = (V\Lambda V^{-1})^n = V\Lambda^n V^{-1}$.

Question 3

(i) Define $p_{ij}(t) = \mathbb{P}(X(t) = j \,|\, X(0) = i)$; then $q_{ij} = p'_{ij}(0)$.
For $j \neq i$ we have the identity $q_{ij} = r_{ij}/\mu_i$, whilst $q_{ii} = -1/\mu_i$.

(ii) a) The problem is that a Markov model is memoryless. If we were to fit a model as suggested, any application in the “letter not written” state would forget what decision had been taken, so an acceptance or rejection letter would be sent completely at random. If we replace “letter not written” by “acceptance letter to be written” and “rejection letter to be written”, the problem goes away.

$$Q = \begin{pmatrix} -1 & 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & -0.1 & 0.08 & 0.02 & 0 & 0 \\ 0 & 0 & -0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & -0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

b) Half the applications only go through the qualification check and the letter-writing, taking an average of 3 days. The other half have to wait for references, so take an average of 13 days. The overall average time is therefore 8 days.
Since applications arrive at a rate of 2 per day on average, this means that there are an average of 16 outstanding at any given time.

c) When the process is in a non-terminal state i we need to generate an exponential r.v. with mean μ_i to simulate the holding time: this is accomplished by setting $H_i = -\mu_i \log U$, where $U \sim U[0,1]$.
In the first two states the destination of the jump is random: state 1 goes to 2 or to 4 with probability 0.5 each; state 2 goes to 3 or to 4 with probabilities 0.8 and 0.2 respectively. In the first case we need a statement like `If U < 0.5 then Destination := 2 otherwise Destination := 4`, in the second similarly.
The variable to track is the time of entry into either of the terminal states. The simulation should be run a large number of times, the proportion of runs when this variable exceeds 20 being the estimate required.

Question 4

(i) There are $\frac{1}{2}N(N-1)$ pairs of students. Of these, $x(N-x)$ involve one student who has heard the rumour and one who has not. The required probability is therefore $\frac{2x(N-x)}{N(N-1)}$.
It follows that $\mathbb{P}(X(t+dt) = x+1 \mid X(t) = x) = \frac{2\lambda x(N-x)}{N(N-1)}\, dt + o(dt)$.

(ii) If X were a Poisson process the probability calculated above would be $\lambda\, dt$; if it were a time-inhomogeneous PP it would be $\lambda(t)\, dt$; for a compound PP it is the size of the jump which is affected, not the time of occurrence. Therefore X does not fit any of these models.
X does, however, possess the Markov property, as the probability of an encounter and the probability that that encounter results in a new recipient of the rumour are both independent of anything except the current value of X.

(iii) If T_x denotes the time until X hits $x+1$ given that it is currently in state x, then we know that T_x is exponentially distributed with mean $\frac{N(N-1)}{2\lambda x(N-x)}$. Therefore the total expected time to hit N starting from 1 is

$$\sum_{x=1}^{N-1} \mathbb{E}(T_x) = \frac{N(N-1)}{2\lambda} \sum_{x=1}^{N-1} \frac{1}{x(N-x)},$$

which can also be written as $\frac{N-1}{\lambda} \sum_{x=1}^{N-1} \frac{1}{x}$.

(iv) The KFE is

$$\frac{d}{dt} p_{1x}(0,t) = -\frac{2\lambda(t) x(N-x)}{N(N-1)} p_{1x}(0,t) + \frac{2\lambda(t)(x-1)(N-x+1)}{N(N-1)} p_{1,x-1}(0,t).$$

The probability required for (a) is $p_{11}(0,t)$, which satisfies $p'_{11}(0,t) = -\frac{2}{N}\lambda(t) p_{11}(0,t)$, with boundary condition $p_{11}(0,0) = 1$, so has solution $p_{11}(0,t) = \exp(-\frac{2}{N}\int_0^t \lambda(u)\, du)$.
For (b) we need to verify that

$$p_{12}(0,t) = \frac{N-1}{N-3}\left\{\exp\left(-\frac{2}{N}\Lambda(t)\right) - \exp\left(-\frac{4(N-2)}{N(N-1)}\Lambda(t)\right)\right\}$$

solves $p'_{12}(0,t) = -\frac{4(N-2)}{N(N-1)}\lambda(t) p_{12}(0,t) + \frac{2}{N}\lambda(t) p_{11}(0,t)$.
The LHS is $-2\frac{N-1}{N(N-3)}\lambda(t)\exp(-2\Lambda(t)/N) + \frac{4(N-2)}{N(N-3)}\lambda(t)\exp(-\frac{4(N-2)}{N(N-1)}\Lambda(t))$.
The RHS is

$$-\frac{4(N-2)}{N(N-1)}\lambda(t)\left\{\frac{N-1}{N-3}\left\{\exp\left(-\frac{2}{N}\Lambda(t)\right) - \exp\left(-\frac{4(N-2)}{N(N-1)}\Lambda(t)\right)\right\}\right\} + \frac{2}{N}\lambda(t)\exp(-\frac{2}{N}\Lambda(t)),$$

which may be written in the form

$$-\frac{2N-2}{N(N-1)}\lambda(t)\exp\left(-\frac{2}{N}\Lambda(t)\right)+\frac{4(N-2)}{N(N-3)}\exp\left(-\frac{4(N-2)}{N(N-1)}\Lambda(t)\right),$$

as required.

Question 5

(i) The history is $\{\mathcal{H}_t : t \geq 0\}$, where $\mathcal{H}_t$ is the collection of all information about the behaviour of X up to time t.
X is a martingale if $\mathbb{E}(X_{t+s} \,|\, \mathcal{H}_t = X_t$.
If X is a non-negative martingale, then there is some limiting variable X_∞ such that $X_t \to X_\infty$ (with probability 1) as $t \to \infty$.
If X is also bounded above then $\mathbb{E}(X_\infty) = X_0$.

(ii) a) Wins arrive in a Poisson stream, so the expected number of wins by time t is λt. Each prize is worth 1. Expenses by time t are $0.02t$. So $\mathbb{E}X(t) = (\lambda - 0.02)t$. In order to make a profit in the long run, the graduate requires $\lambda > 0.02$.

b) $Y(t{+}s) = \exp\left(\alpha(N(t+s) - 0.02(t+s))\right) = Y(t)\exp\left(\alpha(N(t+s) - N(t) - 0.02s)\right)$, where N is the Poisson process describing the incidence of prize wins. Therefore

$$\mathbb{E}(Y(t+s) \,|\, \mathcal{H}_t) = Y(t)e^{-0.02\alpha s}\exp((e^\alpha - 1)\lambda s).$$

This means that Y is a martingale as long as $e^\alpha - 1 = 0.02\alpha/\lambda$.

c) First observe that Y is non-negative, so has an almost sure limit Y_∞.
According to the stopping rule, the stopped version of X can never leave the range $[0, K+1]$. Thus X is bounded above and below, so the same is true of Y.
This means that $\mathbb{E}(Y_\infty) = y_0 = e^{\alpha x_0}$.
But $Y_\infty = e^0 = 1$ with probability θ, say, or somewhere between $e^{\alpha K}$ and $e^{\alpha(K+1)}$ with probability $1-\theta$. It follows that

$$\theta + (1-\theta)e^{\alpha K} \approx e^{\alpha x_0},$$

implying that

$$\theta = \frac{e^{\alpha K} - e^{\alpha x_0}}{e^{\alpha K} - 1}.$$

This is (approx) the probability of hitting zero before hiting K.

The last part requires just that $K \to \infty$. For $\alpha > 0$ it is clear from the above that $\theta \to 1$ when $K \to \infty$. On the other hand, if $\alpha < 0$, then $\theta \to e^{\alpha x_0}$.