Question 1

(i) Discrete time, discrete state space.

(ii) In view of the structure of the process, knowing the value of X_{n-1}, X_{n-2}, ... does not influence $\mathbb{P}(X_{n+1} = y \mid X_n = x)$.
The state space is $\{0, 1, 2, \ldots\}$.

(iii) $\{0\}$ is recurrent, all others transient.

(iv) If $\lambda = \frac{1}{2}$ the size of the population tends to decrease each year, so extinction is ultimately certain.
If $\lambda = 2$ the tendency is for the population to grow, although it is still possible that it will become extinct. If not, the population will expand geometrically.

(v) a) There are two methods: either will do.
• simulate a sequence of exponential random variables T_1, T_2, ... with rate 1 using $T_i = -\log U_i$ and define x as the integer such that $T_1+T_2+\cdots+T_x < \mu < T_1+\cdots+T_{x+1}$ (or $x = 0$ if $T_1 > \mu$).
• Calculate the distribution function F of Poisson(μ) and define x as the integer such that $F(x) < U < F(x + 1)$.

b) In A6 we require `=1+A5`, in B6 `=PoissonVariable(Lambda*B5)`, where we should ensure that the name `Lambda` refers to cell B2.

(vi) Two disadvantages which spring to mind are that the sequence of pseudo-uniform variables is not repeatable and that all variables are recalculated every time a formula is entered.
The Linear Congruential Generator works by taking a seed s_0 then iteratively calculating s_n using $s_n = (as_{n-1} + c)(\text{mod } m)$, where m is a large integer, $0 < a < m$ and $0 \le c < m$. The values returned are $u_n = s_n/m$.
The sequence is entirely determined by the initial seed so is completely repeatable and only needs recalculation when the seed is changed.

Question 2

(i) a) The barriers are both partially reflecting since if $X_n = 0$ there is a possibility, but not a certainty, that $X_{n+1} = 0$, and similarly if $X_n = M$.

b) The probability that a message arrives (resp. is sent) in one time period should be independent of any past events.

(ii) a) $h_k = ph_{k+1} + qh_{k-1} + rh_k$ for $0 < k < M$.
$h_0 = 1$, $h_M = qh_{M-1} + (1 - q)h_M$.

b) This is a difference equation. Auxiliary equation: $px^2 - (p + q)x + q = 0$, with solutions $x = 1$ and $x = q/p$. Therefore the general solution is $h_k = A + B\phi^k$, where $\phi = q/p$.
Applying the boundary conditions, we have $A + B = 1$ and $A + B\phi^M = A + B\phi^{M-1}$, leading to the solution $A = 1$, $B = 0$.
Thus $h_k = 1$ for all k, implying that the chain is certain to hit 0 eventually.

(iii) The number of arrivals in 10^5 time units is 5000, so $\hat{p} = 0.05$.
A suitable estimate for q would be (number of messages delivered)/(number of time periods when the queue was non-empty), but we don't have the information to fill in the denominator.

(iv) X is ergodic because it is irreducible, aperiodic ($p_{00} > 0$) and has finite state space.
DBEs: $\pi_j p_{j,j+1} = \pi_{j+1} p_{j+1,j}$, i.e. $p\pi_j = q\pi_{j+1}$, giving $\pi_j = c\rho^j$, as required, where $\rho = p/q$.
Therefore $c^{-1} = \sum_{j=0}^{M} \rho^j = \frac{1-\rho^{M+1}}{1-\rho}$.

(v) If M is very large and $q > p$, then $\pi_0 \approx 1 - p/q$. Equating this to the observed proportion of the time spent in state 0 gives $1 - \hat{p}/\hat{q} = 0.6$, or $\hat{q} = 2.5\hat{p} = 0.0125$.

Question 3

(i) The assumption is that the probability of promotion or transfer is independent of what happened in past years. This is not entirely reasonable, as a team which has just been relegated is probably more likely to be promoted than one which has been in the lower division for several consecutive years.

(ii) Transition diagram.

(iii) Using the order P, E, W, the transition matrix is

$$P = \begin{pmatrix} 0.8 & 0.16 & 0.04 \\ 0.1 & 0.8 & 0.1 \\ 0.1 & 0.4 & 0.5 \end{pmatrix}.$$

(iv) Chapman-Kolmogorov Equations: $p_{ij}^{(m+n)} = \sum_k p_{ik}^{(m)} p_{kj}^{(n)}$ or, in matrix form, $P^{(m+n)} = P^{(m)} P^{(n)}$.
Since $P^{(1)} = P$ we have $P^{(n)} = PP^{(n-1)} = P^n$.
$P^2 = \begin{pmatrix} .66 & .272 & .068 \\ .17 & .696 & .134 \\ .17 & .536 & .294 \end{pmatrix}$, although the only element we require is 0.272.

(v) We have

$$P\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad P\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1.4 \\ -.7 \\ -.7 \end{pmatrix}, \quad P\begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix} = \begin{pmatrix} 0 \\ .4 \\ -1.6 \end{pmatrix},$$

so that the eigenvalues are 1, 0.7 and 0.4.

(vi) $PV = V\Lambda$, where V is a matrix whose columns are the eigenvectors of P and Λ is a diagonal matrix whose entries are the eigenvalues in the same order. Therefore $P = V\Lambda V^{-1}$ and $P^n = V\Lambda^n V^{-1}$. Finding the nth power of a diagonal matrix is easy.

(vii) The chain should be irreducible and aperiodic.
Both are clearly true here.
To find the limit, solve $\pi^T P = \pi^T$ to get $\pi_P = \frac{5}{15}$, $\pi_E = \frac{8}{15}$, $\pi_W = \frac{2}{15}$. Therefore

$$\lim_{n\to\infty} P^{(n)} = \frac{1}{15}\begin{pmatrix} 5 & 8 & 2 \\ 5 & 8 & 2 \\ 5 & 8 & 2 \end{pmatrix}.$$

Question 4

(i) a) A time series X is called *stationary* if $\mathbb{E}(X_t)$ and $\text{Cov}\,(X_t, X_{t-k})$ do not depend on t and if $\gamma_k \stackrel{def}{=} \text{Cov}\,(X_t, X_{t-k}) \to 0$ as $k \to \infty$.
X is *strictly stationary* if $(X_t, X_{t+1}, \ldots, X_{t+k-1})$ and $(X_1, X_2, \ldots, X_k)$ have the same joint distribution for each $k, t > 0$.

b) The backshift operator, B, acts on a stochastic process X in such a way that $(BX)_t = X_{t-1}$.
Using B notation, MA(1) can be written as $X - \mu = (1 + \beta_1 B)e$, AR(1) as $(1 - \alpha_1 B)(X - \mu) = e$.

c) The condition is that the roots of the characteristic equation $1 - \alpha_1 z - \alpha_2 z^2 - \cdots - \alpha_p z^p$ should all lie outside the unit circle.

d) We need to express the AR(1) in the form $X = (1 + \sum_{k=1}^{\infty} \beta_k B^k)e$. The equation of AR(1) is $(1 - alpha_1 B)X = e$. Inverting this gives

$$X = (1 - \alpha_1 B)^{-1} e = (1 + \alpha_1 B + \alpha_1^2 B^2 + \cdots)e,$$

so that $\beta_k = \alpha_1^k$.

(ii) a) A time series X is *invertible* if it can be expressed in the form of an infinite autoregression, $(1 - \sum_{k=1}^{\infty} \alpha_k B^k)(X - \mu) = e$, such that $\alpha_k \to 0$.

b) A MA is invertible if the roots of $1 + \sum_{k=1}^{q} \beta_q z^q$ all lie outside the unit circle.
In the given case we have $1 - 0.6z - 0.4z^2 = (1 - z)(1 + 0.4z)$, with roots $z = -2.5$ and $z = 1$. One of these roots is not outside the unit circle, so the process is not invertible.

c) $\gamma_0 = (1 + 0.36 + 0.16)\sigma^2 = 1.52\sigma^2$, $\gamma_1 = (-0.6 + 0.24)\sigma^2 = -0.36\sigma^2$, $\gamma_2 = -0.4\sigma^2$ and $\gamma_k = 0$ for $k > 2$.
This gives $\rho_1 = -\frac{0.36}{1.52} = -0.237$, $\rho_2 = -\frac{0.4}{1.52} = -0.263$, $\rho_k = 0$ for $k > 2$.

(iii) a) Y is ARMA(2,0). Its ACF converges to 0 as a sum of geometrically decreasing sequences, but its PACF is equal to zero for lags greater than 2.

b) Z is ARMA(1,1). Both ACF and PACF converge to zero, but neither is actually equal to zero for any lag k.

Question 5

(i) a) For a Poisson process, $N(t+dt) - N(t)$ is equal to 1 with probability $\lambda\, dt + o(dt)$, equal to 0 with probability $1 - \lambda\, dt + o(dt)$, anything else with probability $o(dt)$. This distribution is independent of $\{N(u), u < t\}$ and of $\{N(u) - N(t + dt), u > t + dt\}$.

b) $S(t+dt) = \mathbb{P}(T > t+dt) = \mathbb{P}(T > t)\mathbb{P}(T > t+dt \mid T > t) = S(t)\mathbb{P}(N(t+dt) - N(t) = 0) = S(t)(1 - \lambda\, dt)$.
This implies that $S'(t) = -\lambda S(t)$, with solution $S(t) = S(0)e^{-\lambda t}$.
$S(0) = \mathbb{P}(T > 0) = 1$. $F_T(t) = 1 - S(t) = 1 - e^{-\lambda t}$ and $f_T(t) = \lambda e^{-\lambda t}$.

c) $\mathbb{E}X(t) = 0.2t$.
This shows that the process is non-stationary.

(ii) a) There is clearly a seasonal component with period 6, though not much in the way of a trend. A sensible thing to do would be to look at the seasonally adjusted data.
The equation for the seasonal differencing is $Y_t = X_t - X_{t-6}$.
Seasonal differencing is reasonably sensible, as it does remove seasonal variation from a data set, but it does have its drawbacks. On the other hand, it is part of the Box-Jenkins armoury.

b) Yes, it does seem to be roughly stationary: the SACF is not close to 1 for small lags k. Either MA(1) or AR(1) would be reasonable choices.

c) The residuals appear reasonably uncorrelated, which suggests the AR(1) model is an adequate fit.
A check of the sample partial ACF and/or a normal probability plot of the residuals would be desirable.

d) The fitted process is $Y_t = 8.27 + 0.8228(Y_{t-1} - 8.27) + e_t$.
$\hat{y}_{30}(1) = 8.27 + 0.8228(y_{30} - 8.27)$, $\hat{x}_{30}(1) = \hat{y}_{30}(1) + x_{25}$.