Answer **all four questions** from section A and **any two** of the **four** questions in Section B

SECTION A: COMPULSORY QUESTIONS

Question 1

A discrete-time Markov chain X has finite state space S.

i) Define the terms *irreducible*, *aperiodic*, *stationary distribution* and *limiting distribution*. [4]

ii) If X is irreducible, state whether it possesses a limiting distribution and/or a stationary distribution, distinguishing between the cases where X is aperiodic and where it is periodic. [2]

iv) Let X be a simple random walk with barriers at 0 and K, such that

$$p_{x,x+1} = p \ (0 \le x < K),\ \ p_{K,K} = p,\ \ p_{x,x-1} = q \ (0 < x \le K),\ \ p_{0,0} = q,$$

where $q = 1 - p$.

(a) Show that X is irreducible and aperiodic.

(b) Use the Detailed Balance Equations to determine the limiting distribution of X. [5]

[Total: 11 marks]

Question 2

i) Explain the operation of the Linear Congruential Generator. [3]

ii) Use the inverse distribution function method to derive a method for simulating observations from the exponential density
$f(x) = 5\,e^{-5x}$, $x > 0$. [2]

ii) Describe **one** method of simulating observations from the Gamma density
$h(x) = 25\,x\,e^{-5x}$, $x > 0$. [2]

[Total: 7 marks]

Question 3

A time-homogeneous Markov jump process has states {1, 2, 3} and generator matrix

$$Q = \begin{pmatrix} -6 & 4 & a \\ 0 & b & 3 \\ 0 & 2 & -2 \end{pmatrix}.$$

(i) State the values of the constants *a* and *b*. [1]

(ii) Draw a transition diagram of the process. [1]

(iii) If the process is in state 1 at time 0.5, what is the probability that it is state 1 at time 0.8? [2]

(iv) Calculate the equilibrium distribution of the process. [3]

[Total: 7 marks]

Question 4

State the Chapman-Kolmogorov Equations as they apply to a time-inhomogeneous Markov jump process on a finite state space *S*, and use them to prove the Kolmogorov Forward Equation

$$\frac{\partial}{\partial t} P(s,t) = P(s,t)Q(t).$$

[5 marks]

SECTION B: OPTIONAL QUESTIONS

Question 5

(i) Prove that the distribution of the time until the first jump of a time-homogeneous Poisson process with rate λ is exponential. [4]

(ii) Telephone enquiries are received by an office according to a time-inhomogeneous Poisson process with rate function given by

$$\lambda(t) = 10(2 + \cos t),$$

where t represents the time in hours measured from 9.00am.
Evaluate the conditional probability that exactly 5 calls are received between 10.00am and 10.30am, given that 22 calls were received between 9.00am and 10.00am. [4]

iii) Claims arrive at an insurance office according to a Poisson process $N(t)$ with rate 5 per day. The sizes of the claims, X_i, measured in thousands of pounds, are independent random variables, each having expectation 5 and standard deviation 8. Let $S(t)$ denote the cumulative total of the claims received by time t, i.e.

$$S(t) = \sum_{i=1}^{N(t)} X_i \, .$$

(a) Calculate $E(S(t))$.

(b) The company receives premiums at a constant rate c per day. How large must c be for the company to make a profit on average?

(c) You may assume without proof that $\text{Var}\, S(t) = 5t[\text{Var}(X_i)+(E(X_i))^2]$. If the premiums received by the company total £30,000 per day, use a suitable approximation to find the probability that the premiums received in the first 10 days exceed the claims. [7]

[Total: 15 marks]

Question 6

A discrete-time Markov Chain $\{X_n: n \geq 0\}$ has state space $S = \{\text{L, M, N}\}$ and transition matrix P.

i) For each of the following three cases, draw a transition diagram and mark in transition probabilities such that X satisfies the requirements:

a) States L and M are recurrent and aperiodic, N is transient.

b) X is irreducible and periodic with period 3

c) X is irreducible and periodic with period 2

[6]

ii) In the case

$$P = \begin{pmatrix} 0.6 & 0.3 & 0.1 \\ 0.25 & 0.5 & 0.25 \\ 0.3 & 0.3 & 0.4 \end{pmatrix}$$

show that $PV = V\Lambda$, where Λ is a diagonal matrix and where

$$V = \begin{pmatrix} 1 & -11 & 3 \\ 1 & 10 & -5 \\ 1 & 3 & 3 \end{pmatrix}$$

Hence derive an expression for $\mathbf{P}(X_n = j \mid X_0 = i)$ for each i and j in S and write down the limiting distribution π of the Markov chain X. [9]

[Total: 15 marks]

Question 7

A discrete-time Markov Chain $\{X_n: n \geq 0\}$ has state space $S = \{L, M, N\}$ and transition matrix P.

(i) Give a definition of the terms *random walk* and *simple random walk.* [2]

(ii) An investigation requires a computer simulation of a random walk of the form

$$X_{n+1} = X_n + J_{n+1},$$

where the increments $\{J_n : n = 1, 2, \ldots\}$ each have probability function

$$p(-2) = 0.4, p(0) = 0.3, p(+1) = 0.2, p(+3) = 0.1$$

a) Explain how, given a uniform pseudo-random variable U in the range [0, 1], a simulated value of J could be obtained.

b) An Excel worksheet is constructed with values of $X_0, \ldots, X_{25}$ in cells A5:A30, values of $J_1, \ldots, J_{25}$ in cells B6:B30 and values of $U_1, \ldots, U_{25}$ in cells C6:C30. Assuming that a suitable value of U_{26} in present in cell C31, write down the formulas to put in cells B31 and A31 to generate the next entries. [6]

(iii) The process which is to be modelled is observed for 25 time units, with results as follows:

Time, n	0	1	2	3	4	5	6	7	8	9	10	11	12
Value, X_n	12	14	14	13	14	16	14	13	12	13	12	14	15
Increment, J_n		2	0	−1	1	2	−2	−1	−1	1	−1	2	1
Time, n	13	14	15	16	17	18	19	20	21	22	23	24	25
Value, X_n	18	18	19	17	18	19	22	20	18	16	15	15	17
Increment, J_n	3	0	1	−2	1	1	3	−2	−2	−2	−1	0	1

a) Investigate whether the original model fits the data.

b) Suggest a random walk model which fits the data better than the original model.

c) Suppose you are required to test whether your improved random walk model provides a good fit to the data. State **two** tests which you could perform to determine whether the data appear to come from a random walk. [7]

[Total: 15 marks]

Question 8

When an admissions tutor receives an overseas application form, the procedure is:

- check that the qualifications are acceptable
- if so, obtain reports from referees
- if the references are favourable, accept the application
- send a letter outlining the decision

50% of applicants are found to have unacceptable qualifications and are rejected without obtaining references. Of the remainder, 20% do not receive favourable reports from referees and are also rejected.

Each task takes a random amount of time, the mean times being 1 day for checking qualifications, 10 days for obtaining references and 2 days for sending the letter.

The admissions tutor wishes to model the progress of applications through the system. A continuous-time model is suggested, based on the state space { Awaiting qualifications check, awaiting references, letter not yet written, acceptance letter sent, rejection letter sent }.

i) Explain why the process cannot be represented by a Markov model on this state space but that, by splitting the "letter not written" state into two, a Markov model becomes a possibility. [3]

ii) Draw a graph of the revised state space and mark in all possible direct transitions along with the corresponding transition rates. With the aid of your diagram, write down the generator matrix of the Markov jump process. [4]

iii) Calculate the expectation of the time that passes from the arrival of an application until the letter outlining the decision is sent.
The admissions tutor receives an average of 500 applications per year, roughly evenly distributed over the year. Calculate the average number of applications receiving consideration at any given time. [Assume that a year consists of 250 working days.] [4]

iv) The admissions tutor wishes to find the proportion of applications which take more than 20 days to process. Describe how you would estimate this proportion by simulating this process on a computer. [You may assume that you have a reliable source of pseudo-random numbers uniformly distributed on (0,1).] [4]

[Total: 15 marks]