Stochastic Models Sample paper 2005: Solutions

Question 1

i) Definitions (bookwork)

ii) If aperiodic, there is a limiting distribution.
If periodic, there is a stationary distribution but it is not limiting.

iii) (a) A route such as 0, 1, …, K, K–1, …, 1, 0 is possible, so X is irreducible.
A transition from 0 to 0 is possible, so X is aperiodic.
(b) We have $\pi_i p = \pi_{i+1} q$, so that $\pi_i = (p/q)^i \pi_0$. Since the π_i must sum to 1, we have $\pi_0 = (1-(p/q))/(1-(p/q)^{K+1})$.

Question 2

i) Bookwork.

ii) Simulate expo(5) using T = – 0.2 log U or, equivalently, T = – 0.2 log (1–U).

iii) This is Gamma(2,5), so we can just simulate two expo(5) variables and add them together.

Question 3

i) $a = 2$ and $b = -3$, since rows must sum to 1.

ii) Diagram.

iii) The probability is exp(–6×0.3) = exp(–1.8).

iv) Using $\pi^T Q = 0^T$, we have $\pi_1 = 0$, $3\pi_2 = 2\pi_3$. Since they must sum to 1, this gives $\pi^T = (0, 0.4, 0.6)$.

Question 4
Bookwork

Section B

Question 5

i) Bookwork

ii) First note that the independent increments property holds for time-inhomogeneous Poisson processes, so the number of calls received between 9.00am and 10.00am is irrelevant. The number of calls received between 10.00 and 10.30 is a Poisson r.v., with mean equal to

$$\mu = \int_1^{1.5} 10(2+\cos t)dt = 10(1+\sin 1.5 - \sin 1) = 11.56 .$$

The probability of receiving exactly 5 calls is therefore
$e^{-11.56}(11.56)^5/5! = 0.0164$.

iii) (a) $EX_i = 5$, so ES(t) = 5×5t = 25t, assuming that t is measured in days.
(b) At least 25.
(c) Var S(t) = 5t(64+25) = 445t. After 10 days the company’s net profit has mean £300,000 – 10×£25,000 = £50,000 and variance £4,450m. The probability that the net profit is positive is therefore

$$P\left(Z > \frac{0-50,000}{\sqrt{4,450,000,000}}\right) = 1-\Phi(-0.7495) = 0.7732.$$

Question 6

i) (Diagrams too time-consuming to draw on a computer: transition matrices supplied instead.)

a) L and M recurrent and aperiodic, N transient:

$$P = \begin{pmatrix} a & 1-a & 0 \\ 1-b & b & 0 \\ c & d & 1-c-d \end{pmatrix}$$

for any $0 < a < 1$, $0 < b < 1$, $0 < c+d < 1$.

b) X irreducible with period 3: $P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

c) X irreducible with period 2: $P = \begin{pmatrix} 0 & 1 & 0 \\ 1-a & 0 & a \\ 0 & 1 & 0 \end{pmatrix}$

for any $0 < a < 1$.

ii) Just do the matrix multiplication. We find that the eigenvalues are 1, 0.3 and 0.2. By matrix inversion (Gaussian elimination or whatever) we get

$$V^{-1} = \frac{1}{112}\begin{pmatrix} 45 & 42 & 25 \\ -8 & 0 & 8 \\ -7 & -14 & 21 \end{pmatrix}.$$

This means that $P^n = \frac{1}{112}\begin{pmatrix} 1 & -11 & 3 \\ 1 & 10 & -5 \\ 1 & 3 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.3^n & 0 \\ 0 & 0 & 0.2^n \end{pmatrix}\begin{pmatrix} 45 & 42 & 25 \\ -8 & 0 & 8 \\ -7 & -14 & 21 \end{pmatrix}$,

which can be expanded to give expressions for $P(X_n = j \mid X_0 = i)$. The limiting distribution can be read off the top row of the inverse of V: it is

$$\pi = \left(\frac{45}{112}, \frac{42}{112}, \frac{25}{112}\right)^T.$$

Question 7

i) (a) A random walk is a process $\{X_n: n \geq 0\}$ such that the increments $J_n = X_n - X_{n-1}$ form a sequence of i.i.d. r.v.s.

(b) A simple RW is one where the distribution of the J_n is $p(+1) = p$, $p(-1) = q = 1 - p$.

ii) (a) The easiest way is to use a lookup table, with the cumulative probabilities in the first column and the values in the second, e.g.

0	–2
0.4	0
0.7	1
0.9	3

If this table is named LT, the call =VLOOKUP(Random,LT,2,TRUE) will generate a value from the required distribution.

(b) A suitable formula for B31 is =VLOOKUP(C31,LT,2,TRUE), whereas A31 should contain the formula =A30+B31.

iii) a) Clearly not, since some of the observed increments have zero probability under the suggested distribution.

b) Estimate the distribution of the J_n from the data:

$$\hat{p}(-2) = 0.2,\ \hat{p}(-1) = 0.2,\ \hat{p}(0) = 0.12,\ \hat{p}(1) = 0.28,\ \hat{p}(2) = 0.12,\ \hat{p}(3) = 0.08.$$

c) A test for independence of increments would be one suggestion: a scatter plot of j_{n+1} against j_n should give visual evidence.
It is also a good idea to test whether the increments are independent of the value of the process: a scatter plot of j_{n+1} against x_n will do.

Question 8

i) It is not Markov on the given state space because transitions from the "letter not written" state depend on the previous history of the process: if it arrived in this state directly from the "Awaiting qualifications check" state, then a rejection letter must be sent, but if it arrived from the "Awaiting references" state, then either a rejection or an acceptance letter should be sent.
If the "letter not written" state is split into "Accepted but letter not written" and "Rejected but letter not written", the problem is resolved.

ii) Generator matrix (entries in days^{-1}) is

$$Q = \begin{pmatrix} -1 & 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & -0.1 & 0.08 & 0.02 & 0 & 0 \\ 0 & 0 & -0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & -0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

iii) Every candidate has to go through the phases of evaluating qualifications and waiting for the letter to be written, which take an average of 3 days. 50% of the candidates also have their references checked, taking an average of 10 days. So the overall average is $3 + 0.5 \times 10 = 8$ days.

iv) For half the applications, the time taken can be expressed as
$T = T_1 + T_2 + T_3$, where $T_1 \sim$ expo(1), $T_2 \sim$ expo(0.1) and $T_3 \sim$ expo(0.5).
For the other half, we have $T = T_1 + T_3$. So we can
* take $U_1, \ldots, U_4$ from the set of U[0,1] pseudo-random variables,
* let $V = 0$ if $U_4 < 0.5$ or $V = 1$ if $U_4 > 0.5$,
* set $T = -\log(U_1) - 10\, V \log(U_2) - 2 \log(U_3)$
Do this lots of times and see how often T is observed to be > 20.