Handout 1:
Example of a non convergent Cauchy sequence

Example of a Cauchy sequence over $\mathbb{Q}$ which is not convergent in $\mathbb{Q}$: Recall we showed that the sequence over $\mathbb{Q}$ defined by

$$u_n = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \,. \tag{1}$$

is a Cauchy sequence. We now prove it has no limit in $\mathbb{Q}$ by contradiction. Suppose it converges to a rational number $\frac{a}{b}$ with a, b coprime integers. Let us note the following fact:

$$\begin{aligned} u_{2n+2} - u_{2n} &= \frac{1}{(2n+2)!} - \frac{1}{(2n)!} < 0\,, \\ u_{2n+3} - u_{2n+1} &= \frac{1}{(2n+3)!} - \frac{1}{(2n+1)!} > 0\,. \end{aligned}$$

So the sequence $(u_{2n})_n$ is strictly decreasing and the sequence $(u_{2n+1})_n$ is strictly increasing. And they both converge to $\frac{a}{b}$. Moreover, we have

$$\forall n \in \mathbb{N}, \quad u_{2n} \geq \frac{a}{b}\,,$$

for otherwise, $\exists k \in \mathbb{N}$ such that $\frac{a}{b} - u_{2k} > 0$ so we can use it as an ε. Therefore, $\exists n_0 \in \mathbb{N}$, $n \geq n_0 \Rightarrow |u_{2n} - \frac{a}{b}| < \frac{a}{b} - u_{2k}$. This implies

$$\forall n \geq n_0, \quad u_{2k} < u_{2n}\,,$$

which is in contradiction with $(u_{2n})_n$ being decreasing. Similarly, we have $\forall n \in \mathbb{N}$, $u_{2n+1} \leq \frac{a}{b}$. Therefore, the following holds

$$\forall n \in \mathbb{N}, \quad 0 \leq u_{2n+1} \leq \frac{a}{b} \leq u_{2n}\,.$$

We can assume without loss of generality that $a \geq 0$ and $b > 0$ and we multiply the inequalities by $b(2n)!$ to get

$$\forall n \in \mathbb{N}, \quad b(2n)!u_{2n+1} \leq a(2n)! \leq b(2n)!u_{2n}\,,$$

and then

$$\forall n \in \mathbb{N}, \quad b(2n)!u_{2n+1} - b(2n)!u_{2n} \leq a(2n)! - b(2n)!u_{2n} \leq 0\,.$$

Now

$$b(2n)!u_{2n+1} - b(2n)!u_{2n} = -\frac{b}{2n+1}\,,$$

and there exists a $n_0 \in \mathbb{N}$ such that for $n \geq n_0$

$$-\frac{b}{2n+1} > -1\,.$$

So for all $n \geq n_0$, we have

$$-1 < a(2n)! - b\sum_{k=0}^{2n}(-1)^k\frac{(2n)!}{k!} \leq 0$$

But the quantity between -1 and 0 is an integer, not a rational, so the only possibility is that this quantity is zero. Thus, for any $n \geq n_0$, $u_{2n} = \frac{a}{b}$ *i.e.* , the sequence $(u_{2n})_n$ is constant after the rank n_0, which is in contradiction with it being strictly decreasing. So the sequence $(u_n)_n$ cannot converge to a rational number.