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Factorized Combinations of Virasoro Characters

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Abstract: We investigate linear combinations of characters for minimal Virasoro models which are representable as a product of several basic blocks. Our analysis is based on consideration of asymptotic behaviour of the characters in the quasi-classical limit. In particular, we introduce a notion of the secondary effective central charge. We find all possible cases for which factorization occurs on the base of the Gauß-Jacobi or the Watson identities. Exploiting these results, we establish various types of identities between different characters. In particular, we present several identities generalizing the Rogers–Ramanujan identities. Applications to quasi-particle representations, modular invariant partition functions, super-conformal theories and conformal models with boundaries are briefly discussed.

Introduction

It is a well known fact that the characters of irreducible representations of the Virasoro algebra for the $\mathcal{M}(3,4)$ minimal model possess the peculiar property to be representable as infinite products

$$\chi_{1,2}^{3,4}(q) = q^{\frac{1}{24}} \prod_{n=0}^{\infty} (1+q^{n+1}) = q^{\frac{1}{24}} \prod_{n=0}^{\infty} \left(\frac{1}{1-q^{2n+1}} \right), \tag{0.1}$$

$$\chi_{1,1}^{3,4}(q) \pm \chi_{1,3}^{3,4}(q) = q^{-\frac{1}{48}} \prod_{n=0}^{\infty} (1 \pm q^{n+1/2}). \tag{0.2}$$

As was observed in [1], some characters and linear combinations of characters for the $\mathcal{M}(4,5)$ minimal model admit similar forms.

The question towards a generalization and classification of these identities arises naturally. Surprisingly, it turned out [2] that the only factorizable single characters for minimal models are of type $\chi_{n,m}^{2n,t}(q)$ and $\chi_{n,m}^{3n,t}(q)$. In [3,2,4–6] it was discussed that the factorization of characters in these series is based on the Gauß-Jacobi and Watson identities.

On the other hand, a factorization of linear combinations of Virasoro characters has not been studied so far. In the present paper we show that factorization of combinations $\chi^{s,t}_{n,m}(q) \pm \chi^{s,t}_{n,t-m}(q)$ occurring due to the Gauß-Jacobi and Watson identities is possible (up to the symmetries of the characters) only for $s = 3n, 4n, 6n$. Moreover, we will prove that there are no other factorizable differences of this type which admit the inverse product form similar to the r.h.s. of (0.1).

We present a systematic analysis based on considerations of the asymptotic behaviour of (combinations of) characters in the so-called quasi-classical limit, $q \to 1^-$. We will demonstrate that for linear combinations of the above mentioned type we need, besides the effective central charge $c_{\rm eff}$, the notion of the "secondary" effective central charge $\tilde{c}$.

The advantage to have the characters (or combinations) in the form of infinite products rather than infinite sums is many-fold. First of all the problem of finding the dimension of a particular level in the Verma module of the irreducible representation has been reduced to a simple problem of partitions. As a consequence one may state the possible monomials of Virasoro generators at a specific level. Also the associated quasi-particle states may be constructed from this form without any effort, whereas it is virtually impossible to find them from the infinite sum representation. The quasi-particle form is also related to a classification of Rogers–Ramanujan type of identities [7]. In the present paper this subject is discussed rather briefly in 3.4 and Appendix E. However, this point is followed up further in [31], where the obtained factorized forms of characters were exploited in the derivation of Rogers–Ramanujan type identities. In addition, the factorized characters (or combinations) allow to derive various new identities between different combinations of characters far easier than employing the infinite sum representation. Some of these identities relate different sectors of the same models, whereas others relate different models altogether. Factorized combinations of characters appear naturally in the context of coset models, super-conformal extensions of the Virasoro algebra and boundary conformal field theories. They may even shed some light on massive models, since it was conjectured in [2] that they allow to identify the space of form factors of descendant operators.

1. Preliminaries

We use the notation $\langle n, m\rangle = 1$ if n and m are co-prime numbers and we employ also the standard abbreviation for Euler's function $(q)_m = \prod_{k=1}^{m}(1-q^k)$ with $(q)_0 = 1$.

1.1. Characters of minimal models. The Virasoro algebra is generated by operator valued Fourier coefficients of the energy-momentum tensor $T(z) = \sum_n z^{-n-2} L_n$ and a central charge c. For an irreducible highest weight representation $V_{c,h}$ of the Virasoro algebra with central charge c and weight h one defines the character

$$\chi(q) = \mathrm{tr}_{V_{c,h}} q^{L_0 - \frac{c}{24}} = q^{h-\frac{c}{24}} \sum_{n=0}^{\infty} \mu_n\, q^n, \tag{1.1}$$

with μ_n being the multiplicity of the level n. The corresponding states at a particular level k are spanned by the vectors

$$L_{-k_1} \dots L_{-k_n}|h\rangle, \quad k_1 \le k_2 \le \dots \le k_n, \quad k = \sum_{i=1}^{n} k_i. \tag{1.2}$$

Minimal models are the distinguished conformal theories in which the set of highest weights is finite [8]. These models are labeled by two integer numbers s and t such that

$$s, t \ge 2 \quad \text{and} \quad \langle s, t\rangle = 1. \tag{1.3}$$

The minimal models for which $|s - t| = 1$ are unitary [9,10]. The minimal model $\mathcal{M}(s, t)$ has the central charge

$$c(s, t) = 1 - \frac{6(s-t)^2}{s\ t}. \tag{1.4}$$

The corresponding irreducible highest weight representations of the Virasoro algebra are representations with the weights

$$h^{s,t}_{n,m} = \frac{(nt - ms)^2 - (s-t)^2}{4\ s\ t}\ , \tag{1.5}$$

where the labels run through the following set of integers:

$$1 \le n \le s - 1\ , \quad 1 \le m \le t - 1\ . \tag{1.6}$$

The corresponding character is given by [11,1]

$$\begin{aligned} \chi^{s,t}_{n,m}(q) &= \frac{q^{h^{s,t}_{n,m} - \frac{c(s,t)}{24}}}{(q)_\infty} \sum_{k=-\infty}^{\infty} q^{stk^2} \left(q^{k(nt-ms)} - q^{k(nt+ms)+nm} \right) \\ &= \frac{q^{h^{s,t}_{n,m} - \frac{c(s,t)}{24}}}{(q)_\infty} \hat{\chi}^{s,t}_{n,m}(q), \end{aligned} \tag{1.7}$$

(the second equality defines $\hat{\chi}(q)$, which we refer to as "incomplete character"). The characters possess the following symmetries:

$$\chi^{s,t}_{n,m}(q) = \chi^{t,s}_{m,n}(q) = \chi^{s,t}_{s-n,t-m}(q) = \chi^{t,s}_{t-m,s-n}(q). \tag{1.8}$$

It follows from (1.6) and these symmetries that the minimal model $\mathcal{M}(s, t)$ has $\mathcal{D} = (s-1)(t-1)/2$ different sectors (inequivalent irreducible representations). In addition, (1.7) allows to relate some characters of different models

$$\chi^{\alpha s,t}_{\alpha n,m}(q) = \chi^{s,\alpha t}_{n,\alpha m}(q), \tag{1.9}$$

where α is a positive number such that $\langle s, \alpha t\rangle = \langle t, \alpha s\rangle = 1$. For instance, $\chi^{6,5}_{2,m}(q) = \chi^{3,10}_{1,2m}(q)$.

1.2. Quantum dilogarithm. In our analysis of factorized characters we will be exploiting the properties of the quantum dilogarithm, whose defining relations are

$$\ln_q(\theta) := \prod_{k=0}^{\infty}(1 - e^{2\pi i\theta} q^k) = \exp \sum_{k=1}^{\infty} \frac{1}{k} \frac{e^{2\pi i\theta k}}{q^k - 1}. \tag{1.10}$$

Taking $q = e^{2\pi i\tau}$, we assume that $\mathrm{Im}(\tau) > 0$ and $\mathrm{Im}(\theta) > 0$ in order to guarantee the convergence of (1.10). We see from (1.10) that $\ln_q(\theta)$ is a pseudo-double-periodic function

$$\ln_q(\theta + 1) = \ln_q(\theta) \quad \text{and} \quad \ln_q(\theta + \tau) = \frac{1}{1 - e^{2\pi i\theta}} \ln_q(\theta) \quad . \tag{1.11}$$

It follows easily from this that

$$\ln_q(\theta) = \sum_{k=0}^{\infty} \frac{(-1)^k q^{\frac{k(k-1)}{2}} e^{2\pi i\theta k}}{(q)_k} \qquad \text{and} \qquad \frac{1}{\ln_q(\theta)} = \sum_{k=0}^{\infty} \frac{e^{2\pi i\theta k}}{(q)_k} \quad . \tag{1.12}$$

For explicit calculations it will further turn out to be convenient to employ the notations (in which we will omit the explicit q-dependence as long as q is not varying)

$$\{x\}_y^- := \ln_{q^y}(x\tau) \qquad \text{and} \qquad \{x\}_y^+ := \ln_{q^y}(x\tau + 1/2)\,, \qquad 0 < x \le y. \tag{1.13}$$

These blocks have the following obvious properties:

$$\{x\}_y^+\{x\}_y^- = \{2x\}_{2y}^-\,, \qquad \{x\}_y^\pm = \prod_{k=0}^{n-1}\{x + ky\}_{ny}^\pm, \tag{1.14}$$

$$\ln_{-q^y}(x\tau) = \{x\}_{2y}^-\{x + y\}_{2y}^+\,, \qquad \ln_{-q^y}(x\tau + 1/2) = \{x\}_{2y}^+\{x + y\}_{2y}^-, \tag{1.15}$$

$$\{x\}_{2x}^- = \frac{1}{\{x\}_x^+}\,, \qquad \{x\}_{2x}^+ = \frac{1}{\{x\}_{2x}^-\{2x\}_{2x}^+}\,. \tag{1.16}$$

The last line is Euler's identity which, in fact, can be derived from (1.14).[1]

1.3. Gauß-Jacobi and Watson identities. It will be the principal aim of our manuscript to seek factorizations of some single characters and some linear combinations of characters in the following form:

$$q^{\text{const}} \frac{1}{(q)_\infty} \prod\nolimits_{i=1}^{N} \{x_i\}_y^- \prod\nolimits_{j=1}^{M} \{\tilde{x}_j\}_y^+\,. \tag{1.17}$$

We will encounter the cases $N \neq 0$, $M = 0$ and $N \neq 0$, $M \neq 0$. The explicit formulae of this type, which we will obtain, are based on the Gauß-Jacobi identity (see e.g. [12])

$$\sum_{k=-\infty}^{\infty} (-1)^k v^{\frac{k(k+1)}{2}} w^{\frac{k(k-1)}{2}} = \prod_{k=1}^{\infty}(1-v^k w^{k-1})(1-v^{k-1}w^k)(1-v^k w^k), \tag{1.18}$$

[1] Indeed, using consequently the first and second relation in (1.14) for $y=x$, we obtain $\{2x\}_{2x}^- = \{x\}_x^+\{x\}_x^- = \{x\}_x^+\{x\}_{2x}^-\{2x\}_{2x}^-$, thus deriving the identity $\{x\}_x^+\{x\}_{2x}^- = 1$.

and the Watson identity [13]

$$\sum_{k=-\infty}^{\infty} v^{\frac{3k^2+k}{2}} w^{3k^2}(w^{-2k}-w^{4k+1}) = \prod_{k=1}^{\infty}(1-v^{k-1}w^{2k-1})(1-v^k w^{2k-1}) \times (1-v^k w^{2k})(1-v^{2k-1}w^{4k-4})(1-v^{2k-1}w^{4k}). \tag{1.19}$$

Substituting $v = q^a$, $w = q^b$, we can rewrite the identities in terms of the blocks (1.13)

$$\sum_{k=-\infty}^{\infty} (-1)^k q^{(a+b)\frac{k^2}{2}+\frac{k}{2}(a-b)} = \{a\}^-_{a+b}\{b\}^-_{a+b}\{a+b\}^-_{a+b}, \tag{1.20}$$

$$\sum_{k=-\infty}^{\infty} q^{\frac{3k^2}{2}(a+2b)}\Big(q^{k(a/2-2b)} - q^{k(a/2+4b)+b}\Big) = \{b\}^-_{a+2b}\{a+b\}^-_{a+2b}\{a+2b\}^-_{a+2b} \times\{a\}^-_{2a+4b}\{a+4b\}^-_{2a+4b}. \tag{1.21}$$

Other useful substitutions are $v = q^a$, $w = -q^b$ and $v = -q^a$, $w = q^b$ (for (1.18) it suffices to consider only the first of them, because of the symmetry $v \leftrightarrow w$), which yield

$$\sum_{k=-\infty}^{\infty} (-1)^{\frac{k(k+1)}{2}} q^{(a+b)\frac{k^2}{2}+\frac{k}{2}(a-b)} = \{a\}^-_{2(a+b)}\{b\}^+_{2(a+b)}\{a+b\}^+_{2(a+b)}\{a+2b\}^-_{2(a+b)}\{2a+b\}^+_{2(a+b)}\{2a+2b\}^-_{2(a+b)}, \tag{1.22}$$

$$\sum_{k=-\infty}^{\infty} (-1)^{3k^2} q^{\frac{3k^2}{2}(a+2b)}(q^{k(a/2-2b)} + q^{k(a/2+4b)+b}) = \{b\}^+_{a+2b}\{a+b\}^+_{a+2b}\{a+2b\}^-_{a+2b}\{a\}^-_{2a+4b}\{a+4b\}^-_{2a+4b}, \tag{1.23}$$

$$\sum_{k=-\infty}^{\infty} (-1)^{\frac{k(k-1)}{2}} q^{\frac{3k^2}{2}(a+2b)}(q^{k(a/2-2b)} - q^{k(a/2+4b)+b}) = \{a\}^+_{2a+4b}\{b\}^-_{2a+4b}\{a+b\}^+_{2a+4b} \times \{a+2b\}^+_{2a+4b}\{a+3b\}^+_{2a+4b}\{a+4b\}^+_{2a+4b}\{2a+3b\}^-_{2a+4b}\{2a+4b\}^-_{2a+4b}. \tag{1.24}$$

Here we used (1.15) in order to obtain the r.h.s. in the desired form.

Now one can try to find factorizable linear combinations of characters simply by matching the l.h.s. of (1.20)–(1.24) with appropriate combinations of (1.7). However, this is a cumbersome task. Below we will develop a more systematic and more elegant approach exploiting the quasi-classical asymptotics of characters.

1.4. Quasi-classical asymptotics of characters. As can be seen from (1.10), the limit $\tau \to 0$ of $\ln_q(\theta)$ (since we require $\mathrm{Im}(\tau) > 0$, this is the limit $q \to 1^-$) is singular. The asymptotics is given by

$$\lim_{\tau\to 0} \ln_q(\theta) = \exp\left\{ \frac{1}{2\pi i\tau}\; Li_2\left(e^{2\pi i\theta}\right) + \frac{1}{2}\ln(1-e^{2\pi i\theta}) + \mathcal{O}(\tau) \right\}, \tag{1.25}$$

where $Li_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$ is the Euler dilogarithm [2] (see e.g. [15]).

Introducing $\hat{q} = \exp\{-2\pi i/\tau\}$, we derive from (1.13) and (1.25) the following asymptotics for the limit $q \to 1^-$:

$$\{x\}_y^- \sim \hat{q}^{\frac{Li_2(1)}{4\pi^2 y}} = \hat{q}^{\frac{1}{24y}}, \quad \{x\}_y^+ \sim \hat{q}^{\frac{Li_2(-1)}{4\pi^2 y}} = \hat{q}^{-\frac{1}{48y}}. \tag{1.26}$$

Here we used the fact that $Li_2(1) = -2Li_2(-1) = \pi^2/6$ holds.[3] Notice that $q \to 1^-$ implies that $\hat{q} \to 0^+$, so that $\{x\}_y^-$ and $\{x\}_y^+$ tend to zero and infinity, respectively.

From a physical point of view, say if we regard $\chi(q)$ as a partition function, the limit $\tau \to 0$ can be interpreted as a high-temperature limit (with temperature $T \sim 1/\tau$) which is singular and known to be ruled by the effective central charge only (i.e. it is sector-independent) [16]. Indeed, in order to carry out this limit, one may exploit the behaviour of Virasoro characters under the modular transformation. It is well known [17, 3], that the S-modular transformation ($q \leftrightarrow \hat{q}$) of a character has the following form:

$$\chi_{n,m}^{s,t}(q) = \sum_{n',m'} S_{nm}^{n'm'}\, \chi_{n',m'}^{s,t}(\hat{q}), \tag{1.27}$$

where $S_{nm}^{n'm'}$ are explicitly known constants (see (2.17)). Now it is obvious from (1.7) and (1.27) that

$$\chi_{n,m}^{s,t}(q) \sim S_{nm}^{\bar{n}\bar{m}}\, \hat{q}^{-\frac{c_{\rm eff}(s,t)}{24}} \quad (q \to 1^-). \tag{1.28}$$

Here we have introduced the so-called effective central charge $c_{\rm eff}(s,t) = c(s,t) - 24h_{\bar{n},\bar{m}}^{s,t} = 1 - \frac{6}{st}(\bar{n}t - \bar{m}s)^2$, where $h_{\bar{n},\bar{m}}^{s,t}$ denotes the lowest of all conformal weights in the model. Let us remark that the conditions (1.3) and (1.6) allow us to invoke the well-known theorem of the greatest common divisor and show that $|\bar{n}t - \bar{m}s| = 1$. Hence

$$c_{\rm eff}(s,t) = 1 - \frac{6}{s\,t} \tag{1.29}$$

holds for any minimal model.

Comparison of (1.28) with (1.26) imposes a constraint on the possible structure of characters factorized in form (1.17). Namely, each factor of the type $(\{x\}_y^-)^{\pm 1}$ and $(\{x\}_y^+)^{\pm 1}$ contributes, $\mp\frac{1}{y}$ and $\pm\frac{1}{2y}$ to the effective central charge, respectively. Notice that this is an x independent property. These contributions must sum up to the value given by (1.29).

[2] This motivated the authors of [14] to coin $\ln_q(\theta)$ a quantum dilogarithm.

[3] Equations (1.26) can also be obtained by a saddle point analysis of the identities (1.12) for $\ln_{q^b}(\theta)$ if we put $\theta = x\tau + 1/2$ and $\theta = x\tau$, respectively [6]. In this approach one finds: $\{x\}_y^- \sim \hat{q}^{\frac{L(1)}{4\pi^2 y}}$, $\{x\}_y^+ \sim \hat{q}^{-\frac{L(1/2)}{4\pi^2 y}}$, as $\tau \to 0$. Here $L(z) = Li_2(z) + \frac{1}{2}\ln z \ln(1-z)$ denotes the Rogers dilogarithm [15]. These results coincide with (1.26) since $L(1) = 2L(1/2) = \pi^2/6$.

2. Factorization of Characters

Below it will be useful to refer to the following simple statement:

$$\zeta\, nm + \zeta^{-1} st = nt + ms \quad \Longleftrightarrow \quad t = \zeta m \ \text{ or } \ s = \zeta n. \tag{2.1}$$

Equations of these form will arise as necessary conditions for factorization of (combinations of) characters. Clearly, for s, t, n, m obeying (1.3) and (1.6) the parameter ζ may assume only some rational values greater than unity.

2.1. Factorization of single characters. The factorization of some Virasoro characters in the $\mathcal{M}(3,4)$ and $\mathcal{M}(4,5)$ models was already observed in [1], whereas the factorization of all characters of type $\chi^{2n,t}_{n,m}(q)$ and $\chi^{3n,t}_{n,m}(q)$ was discovered in [2]. It was already discussed in [3,2,4–6] that the factorization of characters in these series may be obtained by exploiting the Gauß-Jacobi and Watson identities. Nevertheless, we wish to present here a systematic derivation of these results based on alternative arguments which will also be applicable in a more general situation.

It is straightforward to see from (1.7) that the first three terms in the expansion of the incomplete character are

$$\hat{\chi}^{s,t}_{n,m}(q) = 1 - q^{nm} - q^{(s-n)(t-m)} + \dots\,, \tag{2.2}$$

and that further terms are of higher powers in q. Let us assume that the incomplete character in question is a particular case of the l.h.s. of the Gauß-Jacobi identity (1.20) for some a and b. Noticing that the series on the l.h.s. of (1.20) is $1 - q^a - q^b +$ higher order terms, we conclude that $a = nm$, $b = (s-n)(t-m)$ or vice versa. Furthermore, the r.h.s. of (1.20) allows to calculate the effective central charge for the character in question. As was explained in Subsect. 1.4, each of the three blocks contributes $-\frac{1}{a+b}$ to c_{eff}. Therefore, $c_{\text{eff}} = 1 - \frac{3}{a+b}$ (the 1 is a contribution of $(q)_\infty = \ln_q(\tau) = \{1\}^-_1$, which appears in (1.7) and whose limit is also ruled by (1.26)). Comparison of this result with (1.29) yields the equation

$$2\,nm + \frac{1}{2}\,st = nt + ms, \tag{2.3}$$

which is a particular case of (2.1) with $\zeta = 2$ and, hence, either $s = 2n$ or $t = 2m$. This implies that $\chi^{2n,t}_{n,m}(q)$ is the only (up to the symmetries (1.8)) possible type of characters factorizable with the help of the Gauß-Jacobi identity and that its factorization has to be of the following form

$$\chi^{2n,t}_{n,m}(q) = \frac{q^{\,h^{2n,t}_{n,m} - \frac{c(2n,t)}{24}}}{(q)_\infty} \{nm\}^-_{nt} \{nt - nm\}^-_{nt} \{nt\}^-_{nt}, \tag{2.4}$$

where t is an odd number according to (1.3). One can verify that Eq. (2.4) is indeed valid by a direct matching of the l.h.s. of (1.20) for the specified a and b with the formula (1.7) for characters (see e.g. [6]).

The same type of consideration applies if we seek characters which are factorizable with the help of the Watson identity. Namely, since the series on the l.h.s. of (1.21) is again $1 - q^a - q^b +$ higher order terms, we conclude that $a = nm$, $b = (s-n)(t-m)$ or vice versa (in contrast to the previous case, these two possibilities lead to different equations).

The r.h.s. of (1.21) allows to calculate the effective central charge: $c_{\text{eff}} = 1 - \frac{4}{y}$, where $y = a + 2b$. Comparison of this value for c_{eff} for the two choices of a and b with (1.29) yields the following equations:

$$\frac{3}{2}nm + \frac{2}{3}st = nt + ms \quad \text{and} \quad 3nm + \frac{1}{3}st = nt + ms, \tag{2.5}$$

respectively. According to (2.1) this implies: $n = 2s/3$ or $m = 2t/3$ in the first case and $n = s/3$ or $m = t/3$ in the second. Notice that these cases are related via the symmetries (1.8). Thus, we conclude that the only possible type of characters factorizable on the base of the Watson identity is $\chi_{n,m}^{3n,t}(q)$ (again up to the symmetries (1.8)) and that its factorization has to be of the following form:

$$\chi_{n,m}^{3n,t}(q) = \frac{q^{h_{n,m}^{3n,t} - \frac{c(3n,t)}{24}}}{(q)_\infty}\{nm\}_{2nt}^{-}\{2nt - nm\}_{2nt}^{-}\{2nt\}_{2nt}^{-}$$
$$\times\{2nt - 2nm\}_{4nt}^{-}\{2nt + 2nm\}_{4nt}^{-}, \tag{2.6}$$

where $\langle 3, t\rangle = 1$. Again, one verifies this formula directly matching it with (1.7) (see [6]).

Thus, we have found all types of characters which are factorizable on the base of the Gauß-Jacobi and the Watson identities. In fact, it was shown in [2] that this exhausts the list of characters of minimal Virasoro models which admit the form (1.17) with M=0 and $x_i \neq x_k$. This implies that for the purpose of factorizing a single character in such a form one does not have to invoke the higher Macdonald identities [18] (also known as the Weyl-Macdonald denominator identities).

As a last remark in this subsection, we notice that in the case $\langle 2, m\rangle = \langle 3, n\rangle = \langle n, m\rangle = 1$ the combination of (2.4) and (2.6) yields

$$\chi_{n,m}^{2n,3m}(q) = \frac{q^{\frac{nm-1}{24}}}{(q)_\infty}\{nm\}_{nm}^{-}. \tag{2.7}$$

The first non-trivial example of this kind is $\chi_{1,2}^{3,4}(q) = q^{\frac{1}{24}}/\{1\}_2^{-} = q^{\frac{1}{24}}\{1\}_1^{+}$ (the second equality is due to the Euler identity). Furthermore, noticing the symmetry $n \leftrightarrow m$ of the r.h.s. of Eq. (2.7), we derive an identity relating different models (it can also be obtained employing (1.9) twice)

$$\chi_{n,m}^{2n,3m}(q) = \chi_{m,n}^{2m,3n}(q), \tag{2.8}$$

where $\langle 6, n\rangle = \langle 6, m\rangle = \langle n, m\rangle = 1$. The first non-trivial example is $\chi_{1,5}^{2,15}(q) = \chi_{1,5}^{3,10}(q)$.

2.2. Factorization of linear combinations. Preliminary ideas. We commence the investigation of factorized linear combinations, $\chi_{n,m}^{s,t}(q) \pm \chi_{n',m'}^{s,t}(q)$, by introducing the quantity

$$\Delta h_{n,m}^{n',m'}(s,t) := h_{n',m'}^{s,t} - h_{n,m}^{s,t}$$
$$= \frac{\big((m+m')s - (n+n')t\big)\big((n-n')t - (m-m')s\big)}{4\,s\,t}, \tag{2.9}$$

where we will often omit the labels s and t. Then

$$\chi^{s,t}_{n,m}(q) \pm \chi^{s,t}_{n',m'}(q) = \frac{q^{h^{s,t}_{n,m} - \frac{c(s,t)}{24}}}{(q)_\infty} \left(\hat{\chi}^{s,t}_{n,m}(q) \pm q^{\,\Delta h^{n',m'}_{n,m}} \hat{\chi}^{s,t}_{n',m'}(q) \right). \tag{2.10}$$

The combination $\hat{\chi}^{s,t}_{n,m}(q) \pm q^{\,\Delta h^{n',m'}_{n,m}} \hat{\chi}^{s,t}_{n',m'}(q)$ can be represented as a product of few blocks $\{\ \}^{\pm}$ only if $\Delta h^{n',m'}_{n,m}$ is an integer or a fraction with sufficiently small denominator (otherwise the product will generate terms with powers of $q^{\Delta h^{n',m'}_{n,m}}$ which are not presented in the combination). On the other hand, the numerator in (2.9) is, in general, not divisible by st because of the conditions (1.3) and (1.6). The only possibility to make this fraction reducible by st is to put $n = n'$ and $m + m' = t$ or, alternatively, $m = m'$ and $n + n' = s$. Thus, we are led to consider the combinations

$$\chi^{s,t}_{n,m}(q) \pm \chi^{s,t}_{n,t-m}(q). \tag{2.11}$$

Let us denote $\Delta h^{n,t-m}_{n,m}(s,t)$ for such pairs by $\Delta h^{s,t}_{n,m}$; its explicit value is

$$\Delta h^{s,t}_{n,m} = \frac{1}{4}(t - 2m)(s - 2n). \tag{2.12}$$

If s or t is even (in particular, this includes all unitary minimal models), then $\Delta h^{s,t}_{n,m}$ is integer or semi-integer. Let, for definiteness, s be even. Then, taking into account the symmetries (1.8), we see that each character in the minimal model $\mathcal{M}(s,t)$ is either of the form $\chi^{s,t}_{s/2,m}(q)$ (i.e., a "single" character, factorizable per se) or there exists exactly one more character such that they form a pair of type (2.11). It follows from this and Eq. (1.6) that the model $\mathcal{M}(s,t)$ has $\mathcal{D}_0 = \frac{t-1}{2}$ "single" characters. Consequently, there are $\mathcal{D}_1 = (\mathcal{D} - \mathcal{D}_0)/2 = \frac{(s-2)(t-1)}{4}$ pairs. If both s and t are odd, then apparently $\mathcal{D}_0 = 0$ and $\mathcal{D}_1 = \mathcal{D}/2 = \frac{(s-1)(t-1)}{4}$.

Consider (2.11) for n and m in the range

$$n < s/2\,, \quad m < t/2. \tag{2.13}$$

For this range all involved characters are different (see e.g. [3]), and it is easy to see that we cover all possible $\mathcal{D}_1$ combinations. Moreover, conditions (2.13) ensure that $\Delta h^{s,t}_{n,m} > 0$. This in turn implies that (2.11) contains only non-negative powers of q. Thus, from now on we will assume that n and m in (2.11) are restricted as in (2.13).

As we have seen in the previous subsection, the knowledge of the asymptotic behaviour of the characters in the $q \to 1^-$ limit proves to be very useful in the search of factorized characters. It turns out that in the case of linear combinations we have to take into account also the next to leading term in (1.28),

$$\chi^{s,t}_{n,m}(q) \;\sim\; S^{\bar{n}\bar{m}}_{nm}\, \hat{q}^{\,-\frac{c_{\text{eff}}(s,t)}{24}} + S^{\tilde{n}\tilde{m}}_{nm}\, \hat{q}^{\,-\frac{\tilde{c}(s,t)}{24}} + \ldots \quad (q \to 1^-), \tag{2.14}$$

where we denoted

$$c_{\text{eff}}(s,t) = c(s,t) - 24\, h^{s,t}_{\bar{n},\bar{m}}\,, \qquad \tilde{c}(s,t) = c(s,t) - 24\, h^{s,t}_{\tilde{n},\tilde{m}}\,. \tag{2.15}$$

Here $h^{s,t}_{\bar{n},\bar{m}}$ and $h^{s,t}_{\tilde{n},\tilde{m}}$ are the smallest and the second smallest conformal weights in the model corresponding to the minimal and the next to minimal value of $|nt - ms|$, respectively. We will refer to $\tilde{c}(s,t)$ as the secondary effective central charge.

As we mentioned above, the theorem of the greatest common divisor ensures that $c_{\rm eff}(s,t) = 1 - 6/st$. Furthermore, one can show in the same way that $|\tilde{n}t - \tilde{m}s| = 2$, so that $h^{s,t}_{\tilde{n},\tilde{m}} = \frac{4-(s-t)^2}{4\,s\,t}$ holds.[4] Thus, employing (1.4) and (2.15), we obtain

$$\tilde{c}(s,t) = 1 - \frac{24}{s\,t}\,. \tag{2.16}$$

The only case where this argument fails is $\mathcal{M}(2,t)$. Here $|\tilde{n}t - \tilde{m}s| = 3$ (unless $t = 3$, in which case $h^{s,t}_{\tilde{n},\tilde{m}}$ does not exist). But, as we demonstrated, in this case all the characters are factorizable per se.

Now, using the explicit form of the matrix S [17,3] involved in the S-modular transformation (1.27)

$$S^{n',m'}_{n,m} = \sqrt{\frac{8}{st}}\,(-1)^{nm'+mn'+1}\,\sin\Bigl(\frac{\pi\, nn't}{s}\Bigr)\sin\Bigl(\frac{\pi\, mm's}{t}\Bigr), \tag{2.17}$$

we observe that $S^{n',m'}_{n,t-m} = -S^{n',m'}_{n,m}(-1)^{n't-m's}$. Taking into account that $|\bar{n}t - \bar{m}s| = 1$ and $|\tilde{n}t - \tilde{m}s| = 2$, we conclude that $S^{\bar{n},\bar{m}}_{n,t-m} = S^{\bar{n},\bar{m}}_{n,m}$ and $S^{\tilde{n},\tilde{m}}_{n,t-m} = -S^{\tilde{n},\tilde{m}}_{n,m}$. Therefore, for the combination $\chi^{s,t}_{n,m}(q) - \chi^{s,t}_{n,t-m}(q)$ the leading terms in (2.14) corresponding to $c_{\rm eff}(s,t)$ cancel but those corresponding to $\tilde{c}(s,t)$ survive[5] and for the combination $\chi^{s,t}_{n,m}(q) + \chi^{s,t}_{n,t-m}(q)$ the leading terms corresponding to $c_{\rm eff}(s,t)$ do not cancel. Thus, we obtain the following asymptotics of the combinations in question:

$$\chi^{s,t}_{n,m}(q) + \chi^{s,t}_{n,t-m}(q) \sim \hat{q}^{\,-\frac{c_{\rm eff}(s,t)}{24}} \quad (q \to 1^-), \tag{2.18}$$

$$\chi^{s,t}_{n,m}(q) - \chi^{s,t}_{n,t-m}(q) \sim \hat{q}^{\,-\frac{\tilde{c}(s,t)}{24}} \quad (q \to 1^-). \tag{2.19}$$

2.3. Factorization of linear combinations. Exact formulae. Now we are in the position to find all combinations of type (2.11) which are factorizable on the base of the Gauß-Jacobi and Watson identities. First, it follows from (1.7) that

$$\hat{\chi}^{s,t}_{n,m}(q) \pm q^{\Delta h^{s,t}_{n,m}}\,\hat{\chi}^{s,t}_{n,t-m}(q) = 1 - q^{nm} \pm q^{\Delta h^{s,t}_{n,m}} + \cdots\,, \tag{2.20}$$

and the further terms are of higher powers in q. Here $\Delta h^{s,t}_{n,m}$ is given by (2.12) and we assume $n < s/2$, $m < t/2$, notice that then $nm \neq \Delta h^{s,t}_{n,m}$

We will consider the sum of characters first. Let us assume that it is factorizable on the base of the Gauß-Jacobi identity (1.22), whose expansion on the l.h.s. is $1 - q^a + q^b +$ higher order terms. Then we infer from (2.20) that $a = nm$ and $b = \Delta h^{s,t}_{n,m}$. The r.h.s. of (1.22) gives the following effective central charge of the combination in question: $c_{\rm eff} = 1 - \frac{3}{4(a+b)}$. Comparing it with (1.29), we obtain the equation $8(a+b) = st$, or more explicitly

$$4\,nm + \frac{1}{4}\,st = nt + ms. \tag{2.21}$$

[4] In fact, a more general statement is valid: for s and t obeying (1.3) and positive integer k such that $k < \min(s,t)$ there exists always a solution of the equation $|nt - ms| = k$ obeying (1.6). It is given by $n = k\bar{n} - ps$ and $m = k\bar{m} - pt$, where p is some integer depending on k.

[5] For $\mathcal{M}(2,t)$ these terms cancel since $|\tilde{n}t - \tilde{m}s| = 3$. This is not surprising because in this case $\chi^{2,t}_{1,m}(q) - \chi^{2,t}_{1,t-m}(q)$ vanishes due to (1.8).

According to (2.1), this implies $4n = s$ or $4m = t$.

If we assume that the difference of characters in (2.20) is factorizable on the base of the Gauß-Jacobi identity (1.20), we have to put $a = nm$, $b = \Delta h^{s,t}_{n,m}$ or vice versa. According to (2.19), the asymptotics $q \to 1^-$ defines the secondary effective central charge, and comparison with the r.h.s. of (1.20) yields $\tilde{c} = 1 - \frac{3}{a+b}$. Together with (2.16), we obtain the equation $8(a + b) = st$ which leads to the same condition (2.21) found for the sum of characters.

Thus, we have shown that the only possible (up to the symmetries (1.8)) combination of characters factorizable on the base of the Gauß-Jacobi identity is $\chi^{4n,t}_{n,m}(q) \pm \chi^{4n,t}_{n,t-m}(q)$ and that its factorization has to be of the following form:

$$\chi^{4n,t}_{n,m}(q) + \chi^{4n,t}_{n,t-m}(q) = \frac{q^{h^{4n,t}_{n,m} - \frac{c(4n,t)}{24}}}{(q)_\infty} \{nm\}^-_{nt} \{nt - nm\}^-_{nt} \{nt\}^-_{nt} \times \{nt/2 - nm\}^+_{nt} \{nt/2\}^+_{nt} \{nt/2 + nm\}^+_{nt}, \tag{2.22}$$

$$\chi^{4n,t}_{n,m}(q) - \chi^{4n,t}_{n,t-m}(q) = \frac{q^{h^{4n,t}_{n,m} - \frac{c(4n,t)}{24}}}{(q)_\infty} \{nm\}^-_{\frac{nt}{2}} \{nt/2 - nm\}^-_{\frac{nt}{2}} \{nt/2\}^-_{\frac{nt}{2}}. \tag{2.23}$$

Here $\langle t, 2\rangle = \langle t, n\rangle = 1$. The direct proof of these relations is performed again by matching them with (1.7) (see Appendix B). Notice that in the case of odd n it suffices to prove only one of the relations, say (2.23). Indeed, in this case $\Delta h^{4n,t}_{n,m} = n(t - 2m)/2$ is semi-integer, so that changing the signs of all semi-integer powers in the series on the l.h.s. of (2.22), we obtain the series on the l.h.s. of (2.23). Therefore, the r.h.s. of (2.22) is derived from the r.h.s. of (2.23) with the help of (1.15).

Now we apply the same technique as above in order to find the differences of type (2.11) which are factorizable on the base of the Watson identity. We assume they have the form of Eq. (1.21), whose expansion on the l.h.s. is $1 - q^a - q^b +$ higher order terms. Then we infer from (2.20) that $a = nm$ and $b = \Delta h^{s,t}_{n,m}$ or $a = \Delta h^{s,t}_{n,m}$ and $b = nm$. The r.h.s. of (1.21) gives the following secondary effective central charge of the combination in question: $\tilde{c} = 1 - \frac{4}{a+2b}$. Comparing it with (2.16), we obtain the equation $6(a + 2b) = st$, which gives for the two possible choices of a and b,

$$3\,nm + \frac{1}{3}\,st = nt + ms, \quad \text{and} \quad 6\,nm + \frac{1}{6}\,st = nt + ms, \tag{2.24}$$

respectively. According to (2.1), this implies $3n = s$ or $3m = t$ in the first case and $6n = s$ or $6m = t$ in the second.

Assuming that the sum of characters in (2.20) is factorized on the base of the Watson identity (1.23), we have to put $a = nm$, $b = \Delta h^{s,t}_{n,m}$. Then the r.h.s. of (1.23) gives $c_{\text{eff}} = 1 - \frac{1}{a+2b}$. Comparing it with (1.29), we obtain the equation $6(a + 2b) = st$ and, thus, we recover the first equation in (2.24). So, this is once more the case $n = s/3$ or $m = t/3$.

It turns out that the sum of characters in (2.20) cannot be factorized on the base of the Watson identity (1.24). Indeed, its l.h.s. is the following series $1 + q^a - q^b + q^{2a+b} - q^{a+5b} +$ higher order terms. On the other hand, for $n < s/2$, $m < t/2$ we have

$$\hat{\chi}^{s,t}_{n,m}(q) \pm q^{\Delta h^{s,t}_{n,m}} \hat{\chi}^{s,t}_{n,t-m}(q) = 1 - q^{nm} \pm q^{\Delta h^{s,t}_{n,m}} \mp q^{\Delta h^{s,t}_{n,m} + n(t-m)} \mp q^{\Delta h^{s,t}_{n,m} + m(s-n)} + \dots, \tag{2.25}$$

where further terms are of higher powers in q. Evidently, these two series cannot match because of the wrong sign of the q^{2a+b} term.

Thus, the only possible (up to the symmetries (1.8)) combinations of characters factorizable on the base of the Watson identity are $\chi^{3n,t}_{n,m}(q) \pm \chi^{3n,t}_{n,t-m}(q)$ and $\chi^{6n,t}_{n,m}(q) - \chi^{6n,t}_{n,t-m}(q)$ and their factorizations have to be of the following form

$$\chi^{3n,t}_{n,m}(q) \pm \chi^{3n,t}_{n,t-m}(q) = \frac{q^{h^{3n,t}_{n,m} - \frac{c(3n,t)}{24}}}{(q)_\infty} \{nm\}^-_{nt} \{nt - nm\}^-_{nt} \times \left\{\frac{nt}{2}\right\}^-_{\frac{nt}{2}} \left\{\frac{nt-2nm}{4}\right\}^\pm_{\frac{nt}{2}} \left\{\frac{nt+2nm}{4}\right\}^\pm_{\frac{nt}{2}}, \tag{2.26}$$

$$\chi^{6n,t}_{n,m}(q) - \chi^{6n,t}_{n,t-m}(q) = \frac{q^{h^{6n,t}_{n,m} - \frac{c}{24}(6n,t)}}{(q)_\infty} \{nm\}^-_{nt} \{nt - nm\}^-_{nt} \{nt\}^-_{nt} \times \{nt - 2nm\}^-_{2nt} \{nt + 2nm\}^-_{2nt}. \tag{2.27}$$

Here $\langle t, 3\rangle = \langle t, n\rangle = 1$ in (2.26) and $\langle t, 6\rangle = \langle t, n\rangle = 1$ in (2.27). The direct proof of these relations is performed again by matching them with (1.7) (see Appendix B).

Combining (2.22)–(2.23) and (2.26), we also obtain

$$\chi^{4n,3m}_{n,m}(q) + \chi^{4n,3m}_{n,2m}(q) = \frac{q^{\frac{nm-2}{48}}}{(q)_\infty} \{nm\}^-_{nm} \{nm/2\}^+_{nm}, \tag{2.28}$$

$$\chi^{4n,3m}_{n,m}(q) - \chi^{4n,3m}_{n,2m}(q) = \frac{q^{\frac{nm-2}{48}}}{(q)_\infty} \{nm/2\}^-_{\frac{nm}{2}}, \tag{2.29}$$

where $\langle n, 3\rangle = \langle m, 2\rangle = \langle n, m\rangle = 1$.

To conclude this subsection we mention an interesting byproduct, which follows from (2.6) and (2.26),

$$\begin{aligned} &\{nm\}^-_{nt}\, \{nt - nm\}^-_{nt} \left\{\frac{nt}{2}\right\}^-_{\frac{nt}{2}} \left\{\frac{nt-2nm}{4}\right\}^\pm_{\frac{nt}{2}} \left\{\frac{nt+2nm}{4}\right\}^\pm_{\frac{nt}{2}} \\ &\quad = \{2nt\}^-_{2nt}\, \{nm\}^-_{2nt}\, \{2nt - nm\}^-_{2nt}\, \{2nt - 2nm\}^-_{4nt}\, \{2nt + 2nm\}^-_{4nt} \\ &\quad \pm q^{\frac{nt-2nm}{4}}\, \{2nt\}^-_{2nt}\, \{nt - nm\}^-_{2nt}\, \{nt + nm\}^-_{2nt}\, \{2nm\}^-_{4nt}\, \{4nt - 2nm\}^-_{4nt}\,, \end{aligned} \tag{2.30}$$

which may also be rewritten as

$$\begin{aligned} &\left\{\frac{nt}{2}\right\}^-_{nt} \left\{\frac{nt-2nm}{4}\right\}^\pm_{\frac{nt}{2}} \left\{\frac{nt+2nm}{4}\right\}^\pm_{\frac{nt}{2}} \\ &= \{nt\}^+_{nt} \{nt - nm\}^+_{2nt} \{nt + nm\}^+_{2nt} \pm q^{\frac{nt-2nm}{4}}\, \{nt\}^+_{nt} \{nm\}^+_{2nt} \{2nt - nm\}^+_{2nt}. \end{aligned} \tag{2.31}$$

This identity resembles particular formulae in [19] ((A5) and (A6) therein), which were useful to derive a different type of identities between characters.

Analogous identities following from (2.28)–(2.29) and (2.26) are

$$\begin{aligned} &\{3nm\}^+_{8nm} \{5nm\}^+_{8nm} \pm q^{\frac{nm}{2}} \{nm\}^+_{8nm} \{7nm\}^+_{8nm} \\ &= \{nm/2\}^\pm_{nm} \{2nm\}^-_{4nm} \{4nm\}^-_{8nm}. \end{aligned} \tag{2.32}$$

2.4. Remarks on the factorized combinations. The factorized characters given by (2.4) and (2.6) can be rewritten in the "inverse product" form (examples of such representation are given in Appendix A)[6]

$$q^{\mathrm{const}} \frac{1}{\prod_{i=1}^{N} \{x_i\}_{y_i}^{-}} \,. \tag{2.33}$$

In order to achieve this, one rewrites $(q)_\infty = \{1\}_1^-$ with the help of (1.14) as a product of some number of blocks and then cancels all blocks in the numerator with some of those in the denominator. The only problem here is to verify that all blocks in (2.4) and (2.6) are different. Equation (2.4) could have coinciding blocks only if $t = 2m$. This is however excluded by the condition $\langle t, 2\rangle = 1$ which must hold because of (1.3). Equation (2.6) could have coinciding blocks if $t = 3m$, $t = 3m/2$ or $t = 2m$. The first two possibilities are excluded by the condition $\langle t, 3\rangle = 1$. The last one is allowed, but this case is described by the reduced formula (2.7), which is obviously representable in the form (2.33).

The inverse product form (2.33) (it is rather common for characters of Kac-Moody algebras [12]) can be interpreted as a character of a module with states created by bosonic type operators. Having the characters in the form (2.33) implies that the dimension of the level k in the Verma module of the irreducible representation is simply the number of partitions $k = x_1 + \ldots + x_N + \sum_{i=1}^{N} n_i y_i$ with n_i being an arbitrary non-negative integer. This suggests that the states at this level are simply monomials of the form (1.2). If any power of a generator having a given grading k is allowed, the character acquires a factor $(1 - q^k)$ in the denominator. It is guaranteed that any monomial by itself (apart from $L_{-1}|h = 0\rangle$) can never constitute a null-vector, as follows from the following simple argument. A null-vector has by definition zero norm or equivalently it is annihilated by L_n for all $n > 0$. Hence to prove our statement it is sufficient to show for one n that L_n acting on (1.2) is non-vanishing. It is easy to verify for $k_1 \neq k$ that L_k acting on (1.2) vanishes only for $h = 0$. In case $k_1 = k \neq 1$, the action of L_{k-1} is always non-vanishing. However, one may not guarantee that all these monomials are linearly independent.

It turns out that all of the factorized combinations of characters (2.22)–(2.27) and (2.28)–(2.29) can be rewritten in the inverse product form generalizing (2.33), namely as

$$q^{\mathrm{const}} \frac{1}{\prod_{i=1}^{N} \{x_j\}_{y_j}^{-} \ \prod_{j=1}^{M} \{\tilde{x}_i\}_{y_i}^{+}} \,. \tag{2.34}$$

In particular, (2.23) for even n, the lower sign in (2.26) for integer $nt/4$ and $nm/2$, and (2.27) can be analyzed easily in the way we presented above and correspond to (2.34) with $M = 0$. The analysis of other cases is slightly more involved (since we encounter $\{\ \}^+$ blocks and blocks with non-integer arguments) but goes essentially along the same lines. Consider, for instance, (2.22). Using (1.14) and (1.16), we can rewrite its r.h.s. as follows (we use here the notation $\{x_1; \ldots ; x_n\}_y^{\pm} := \{x_1\}_y^{\pm} \ldots \{x_n\}_y^{\pm}$)

$$q^{\mathrm{const}} \frac{\{nm; n(t-2m); n(t-m); nt; n(t+m); n(2t-m); n(t+2m); 2nt\}_{2nt}^{-}}{\{1\}_1^{-}\ \{nt/2 - nm; nt/2; nt/2 + nm)\}_{nt}^{-}\ \{nt\}_{nt}^{+}} \,.$$

For n and m in the range (2.13) the numerator could have coinciding blocks only if $t = 3m$. However, in this case we have the reduced formula (2.28) which is readily seen

[6] Exactly this form was an aim in [2].

to be representable in form (2.34) if we take (1.16) into account. Analysis of (2.23), (2.26) and (2.29) is performed analogously (notice only that for (2.26) one has to distinguish the cases $nt/2 = 0, 1 \bmod 2$).

Thus, all the factorizable combinations of characters of type (2.11) admit the form (2.34). Examples of such representation are given in Appendix A. Moreover, we prove (see Appendix C) that there are no other factorizable differences of this type which admit the inverse product form (2.33). This is a rather surprising fact because the Gauß-Jacobi and Watson identities are the specific Macdonald identities [18] for the $A_1^{(1)}$ and $A_2^{(2)}$ algebras and one could expect that the higher Macdonald identities also lead to similar factorizations.

It is worth to notice that some of the factorizable combinations discussed above admit the following form

$$q^{\mathrm{const}} \, \frac{\prod_{j=1}^{M} \{\tilde{x}_i\}^+_{y_i}}{\prod_{i=1}^{N} \{x_j\}^-_{y_j}} \,. \tag{2.35}$$

This is the most natural form if we consider such an expression as a character (e.g. in the context of the super-conformal models, see Subsect. 3.4) of a module with states created not only by bosonic type operators but also by fermionic type operators, which produce the blocks in the numerator. Also, the form (2.35) gives particularly simple formulae for quasi-particle momenta (see Subsect. 3.3).

3. Applications

In the rest of the paper we will present some corollaries and applications of the obtained results both in a mathematical and physical context.

3.1. Identities between characters. We commence by matching the product sides of the formulae for the factorized linear combinations of characters with those for the factorized single characters. For (2.23) this yields

$$\chi^{8n,t}_{2n,m}(q) - \chi^{8n,t}_{2n,t-m}(q) \;=\; \chi^{2n,t}_{n,2m}(q), \tag{3.1}$$

where $\langle t, 2\rangle = \langle t, n\rangle = 1$. Notice that this identity is exact in the sense that is it does not need an extra factor of type q^{const} on the r.h.s. because $h^{2n,t}_{n,2m} - c(2n,t)/24 = h^{8n,t}_{2n,m} - c(8n,t)/24$.[7] Since $\chi^{2,3}_{1,1}(q) = 1$, we obtain, as a particular case, the identity (which was also presented in [3] in a different context)

$$\chi^{3,8}_{1,2}(q) - \chi^{3,8}_{1,6}(q) = 1.$$

This is the only possible identity of the type $\chi^{s,t}_{n,m}(q) - \chi^{n,t}_{s,t-m}(q) = q^{\mathrm{const}}$ because it requires $\tilde{c}(s,t) = 0$. According to (2.16), this implies $st = 24$. The latter equation is solved uniquely (up to a permutation of s and t) due to (1.3).

For (2.27) and (2.26) we obtain analogously

$$\chi^{12n,t}_{4n,m}(q) - \chi^{12n,t}_{8n,m}(q) \;=\; \chi^{3n,t}_{2n,2m}(q), \quad \chi^{12n,t}_{2n,m}(q) - \chi^{12n,t}_{10n,m}(q) \;=\; \chi^{3n,t}_{n,2m}(q), \tag{3.2}$$

[7] This property, which actually holds for all identities in this subsection, hints on specific modular properties of the combinations of type (2.11).

where $\langle t, 6\rangle = \langle n, t\rangle = 1$. These identities are also exact. The first nontrivial examples of this kind are $\chi^{12,5}_{4,m}(q) - \chi^{12,5}_{8,m}(q) = \chi^{3,5}_{2,2m}(q)$ and $\chi^{12,5}_{2,m}(q) - \chi^{12,5}_{10,m}(q) = \chi^{3,5}_{1,2m}(q)$, $m = 1, 2$. Furthermore, the characters on the r.h.s. of (3.2) form a pair of the type (2.11), and applying (2.26), we obtain (assuming $m < t/4$ for definiteness)

$$\begin{aligned}&\chi^{12n,t}_{2n,m}(q) - \chi^{12n,t}_{10n,m}(q) \pm \chi^{12n,t}_{4n,m}(q) \mp \chi^{12n,t}_{8n,m}(q) \\ &= \frac{q^{h^{3n,t}_{n,2m} - \frac{c(3n,t)}{24}}}{(q)_\infty}\{2nm\}^-_{nt}\{nt-2nm\}^-_{nt}\{nt/2\}^-_{\frac{nt}{2}}\{nt/4-nm\}^\pm_{\frac{nt}{2}}\{nt/4+nm\}^\pm_{\frac{nt}{2}}.\end{aligned} \tag{3.3}$$

Finally, matching the r.h.s. of (3.3) for the lower sign with (2.6), we obtain

$$\chi^{48n,t}_{8n,m}(q) - \chi^{48n,t}_{16n,m}(q) + \chi^{48n,t}_{32n,m}(q) - \chi^{48n,t}_{40n,m}(q) = \chi^{3n,t}_{2n,4m}(q). \tag{3.4}$$

The first nontrivial example is $\chi^{48,5}_{8,1}(q) - \chi^{48,5}_{16,1}(q) + \chi^{48,5}_{32,1}(q) - \chi^{48,5}_{40,1}(q) = \chi^{3,5}_{1,1}(q)$.

Another way to derive some new identities is to match the product sides of different factorized linear combinations. In particular, one easily recovers the property (1.9) for combinations

$$\chi^{s,\alpha t}_{n,\alpha m}(q) \pm \chi^{s,\alpha t}_{n,\alpha(t-m)}(q) = \chi^{\alpha s,t}_{\alpha n,m}(q) \pm \chi^{\alpha s,t}_{\alpha n,t-m}(q), \tag{3.5}$$

if α is a positive integer such that $\langle t,\alpha\rangle = \langle s,\alpha\rangle = 1$. For instance, $\chi^{3,10}_{1,2m}(q) \pm \chi^{3,10}_{2,2m}(q) = \chi^{5,6}_{m,2}(q) \pm \chi^{5,6}_{m,4}(q)$, $m = 1, 2$.

Less obvious identities between characters of different models having the same c_{eff} follow if we compare the r.h.s. of (2.26) and (2.27):

$$\chi^{3n,2t}_{n,t-2m}(q) - \chi^{3n,2t}_{n,t+2m}(q) = \chi^{6n,t}_{n,m}(q) - \chi^{6n,t}_{5n,m}(q), \tag{3.6}$$

where $\langle t, 6\rangle = \langle n, 2\rangle = \langle t, n\rangle = 1$, $m < t/2$. For instance, $\chi^{3,10}_{1,1}(q) - \chi^{3,10}_{2,1}(q) = \chi^{5,6}_{2,1}(q) - \chi^{5,6}_{2,5}(q)$.

Employing the factorized form of (combinations of) characters, we can derive identities involving their bilinear combinations. For instance, it is straightforward to verify the following relations (see Appendix D for a sample proof)

$$\chi^{3n,2m}_{n,m}\chi^{4n,5m}_{2n,m} = \chi^{3n,4m}_{n,2m}\left(\chi^{6n,5m}_{n,2m} - \chi^{6n,5m}_{n,3m}\right), \tag{3.7}$$

$$\chi^{3n,2m}_{n,m}\chi^{4n,5m}_{2n,2m} = \chi^{3n,4m}_{n,2m}\left(\chi^{6n,5m}_{n,m} - \chi^{6n,5m}_{n,4m}\right), \tag{3.8}$$

$$\chi^{3n,2m}_{n,m}\left(\chi^{4n,5m}_{n,m} \pm \chi^{4n,5m}_{n,4m}\right) = \left(\chi^{6n,5m}_{2n,2m} \mp \chi^{6n,5m}_{2n,3m}\right)\left(\chi^{3n,4m}_{n,m} \pm \chi^{3n,4m}_{n,3m}\right), \tag{3.9}$$

$$\chi^{3n,2m}_{n,m}\left(\chi^{4n,5m}_{n,2m} \pm \chi^{4n,5m}_{n,3m}\right) = \left(\chi^{6n,5m}_{2n,m} \mp \chi^{6n,5m}_{2n,4m}\right)\left(\chi^{3n,4m}_{n,m} \pm \chi^{3n,4m}_{n,3m}\right), \tag{3.10}$$

which in turn lead to the identities

$$\chi^{4n,5m}_{2n,m}\left(\chi^{6n,5m}_{n,m} - \chi^{6n,5m}_{n,4m}\right) = \chi^{4n,5m}_{2n,2m}\left(\chi^{6n,5m}_{n,2m} - \chi^{6n,5m}_{n,3m}\right), \tag{3.11}$$

$$\begin{aligned}&\left(\chi^{4n,5m}_{n,2m} \pm \chi^{4n,5m}_{n,3m}\right)\left(\chi^{6n,5m}_{2n,2m} \mp \chi^{6n,5m}_{2n,2m}\right) \\ &\qquad = \left(\chi^{4n,5m}_{n,m} \pm \chi^{4n,5m}_{n,4m}\right)\left(\chi^{6n,5m}_{2n,m} \mp \chi^{6n,5m}_{2n,4m}\right).\end{aligned} \tag{3.12}$$

We have omitted the q-dependence for compactness of the formulae. Once more we like to point out these relations are exact (see (D.1)). A particular case of (3.11) and (3.12) for $n = m = 1$ was found in [19]. Further interesting identities are for instance

$$\begin{aligned}&\left(\chi_{n,m}^{3n,4m}(q) + \chi_{n,3m}^{3n,4m}(q)\right)\left(\chi_{n,m}^{3n,4m}(q) - \chi_{n,3m}^{3n,4m}(q)\right)\\ &\quad = q^{-\frac{nm}{24}}\left(\chi_{n,m}^{3n,2m}(q)\right)^2 \{nm\}_{2nm}^{-},\end{aligned} \tag{3.13}$$

$$\chi_{1,2}^{5,6}(q)\chi_{2,2}^{5,6}(q) - \chi_{1,4}^{5,6}(q)\chi_{2,4}^{5,6}(q) = \left(\chi_{1,2}^{3,4}(q)\right)^2, \tag{3.14}$$

$$\left(\chi_{1,2}^{4,5}(q) \pm \chi_{1,3}^{4,5}(q)\right)\left(\chi_{2,2}^{5,6}(q) \mp \chi_{2,4}^{5,6}(q)\right) = \chi_{1,5}^{4,15}(q) \pm \chi_{3,5}^{4,15}(q). \tag{3.15}$$

Equation (3.13) for $n = m = 1$ yields the well-known relation

$$\left((\chi_{1,1}^{3,4}(q))^2 - (\chi_{1,3}^{3,4}(q))^2\right)\chi_{1,2}^{3,4}(q) = 1$$

It is of a certain interest to search for relations between (combinations of) characters with rescaled q. The rescaling, $q \to q^r$ or, equivalently, $\tau \to r\tau$ can be regarded as a transformation relating theories on two different tori. In statistical mechanics, where τ is considered as a physical parameter (e.g. inverse temperature or width of a strip), this transformation relates two models at different values of this parameter.

In order to match the factorized (combinations of) characters involving those with rescaled q it is useful to take into account that such rescaling, $q \to q^r$, also leads to the rescaling of $c_{\text{eff}} \to c_{\text{eff}}/r$ (and $\tilde{c} \to \tilde{c}/r$). We present here only several examples relating characters of some models with interesting physical content under the transformation $q \to q^2$:

$$\chi_{1,1}^{3,4}(q^2) - \chi_{1,3}^{3,4}(q^2) = \left(\chi_{1,2}^{3,4}(q)\right)^{-1}, \tag{3.16}$$

$$\chi_{1,2}^{5,6}(q^2) + \chi_{1,4}^{5,6}(q^2) = \chi_{1,1}^{2,5}(q), \quad \chi_{2,2}^{5,6}(q^2) + \chi_{2,4}^{5,6}(q^2) = \chi_{1,2}^{2,5}(q), \tag{3.17}$$

$$\chi_{1,1}^{5,6}(q) - \chi_{1,5}^{5,6}(q) = \chi_{1,2}^{2,5}(q^2), \quad \chi_{2,1}^{5,6}(q) - \chi_{2,5}^{5,6}(q) = \chi_{1,1}^{2,5}(q^2), \tag{3.18}$$

$$\chi_{2,1}^{6,7}(q^2) + \chi_{2,6}^{6,7}(q^2) = \chi_{1,3}^{6,7}(q) - \chi_{1,4}^{6,7}(q), \tag{3.19a}$$

$$\chi_{2,2}^{6,7}(q^2) + \chi_{2,5}^{6,7}(q^2) = \chi_{1,1}^{6,7}(q) - \chi_{1,6}^{6,7}(q), \tag{3.19b}$$

$$\chi_{2,3}^{6,7}(q^2) + \chi_{2,4}^{6,7}(q^2) = \chi_{1,2}^{6,7}(q) - \chi_{1,5}^{6,7}(q). \tag{3.20}$$

Finally, it may be of some interest to consider relations between (combinations of) incomplete characters with rescaled q. For instance, we have

$$\hat{\chi}_{n,m}^{4n,t}(q^2) - \hat{\chi}_{3n,m}^{4n,t}(q^2) = \hat{\chi}_{n,2m}^{2n,t}(q), \quad \hat{\chi}_{n,m}^{6n,t}(q^2) - \hat{\chi}_{5n,m}^{6n,t}(q^2) = \hat{\chi}_{n,2m}^{3n,t}(q). \tag{3.21}$$

Identities between the corresponding full characters are then obtained by multiplication of the r.h.s. with $(q^{\frac{1}{24}}\{1\}_1^{+})^{-1}$.

3.2. Rogers-Ramanujan type identities. Once we have achieved factorization of (combinations of) characters in the form (2.35), we can employ (1.12) in order to re-express the product as a sum of type distinct from (1.7). More precisely, combining (1.12) with (1.13) and substituting into (2.35), we obtain

$$q^{\text{const}} \frac{\prod_{j=1}^{M}\{\tilde{x}_i\}_y^+}{\prod_{i=1}^{N}\{x_j\}_y^-} = \sum_{\boldsymbol{l}} \frac{q^{(l_1^2+\ldots+l_M^2-l_1-\ldots-l_M)y/2+\boldsymbol{B}\cdot\boldsymbol{l}}}{(q^y)_{l_1}\ldots(q^y)_{l_{M+N}}}, \tag{3.22}$$

where $\boldsymbol{B} = \{\tilde{x}_1, \ldots \tilde{x}_M, x_1, \ldots x_N\}$ and $\boldsymbol{l}$ has $(M+N)$ components running through non-negative integers. The structure of this identity resembles the famous Rogers–Ramanujan identities (which are in fact just two ways of writing down the characters $\chi_{1,1}^{2,5}$ and $\chi_{1,2}^{2,5}$ – see Appendix A)

$$\frac{1}{\{1;4\}_5^-} = \sum_{l=0}^{\infty} \frac{q^{l^2}}{(q)_l}, \qquad \frac{1}{\{2;3\}_5^-} = \sum_{l=0}^{\infty} \frac{q^{l^2+l}}{(q)_l}\ . \tag{3.23}$$

However, whereas Eq. (3.22) may be decomposed into a product of identities (1.12), such simplifications are not possible in the proof of the Rogers–Ramanujan identities (see e.g. [23]). Thus, in order to obtain more interesting generalizations of the Rogers-Ramanujan identities involving our factorized form of (combinations of) characters as a product side, we need another expression for the sum on the r.h.s. of (3.22). For this purpose we make use of the results of [21] where it was observed that some Virasoro characters admit the following form:

$$q^{\text{const}} \sum_{\boldsymbol{l}} \frac{q^{\boldsymbol{l}^t A \boldsymbol{l} + \boldsymbol{B}\cdot\boldsymbol{l}}}{(q)_{l_1}\ldots(q)_{l_n}}\ , \tag{3.24}$$

where A is a real $n \times n$ symmetric matrix (sometimes coinciding with the inverse Cartan matrix of a simply-laced Lie algebra), $\boldsymbol{B}$ is an n-component vector, and the summation may be restricted by a condition of the type $\boldsymbol{\gamma}\cdot\boldsymbol{l} = Q \ (\text{mod}\,\alpha)$ with some integer valued $\boldsymbol{\gamma}$ and positive Q and α. It turns out that some of the characters of minimal models admitting the form (3.24) are either factorizable per se or can be combined into the factorizable combinations considered above. This circumstance allows us to apply the results of Sect. 2 and derive a set of Rogers–Ramanujan type identities. For instance

$$q^{-\frac{1}{40}}\chi_{1,1}^{3,5}(q) = \sum_{\substack{l=0\\ \text{even}}}^{\infty} \frac{q^{(l^2+2l)/4}}{(q)_l} = \frac{\{4\}_{10}^+\{6\}_{10}^+}{\{2\}_{10}^-\{3\}_{10}^-\{5\}_{10}^-\{7\}_{10}^-\{8\}_{10}^-}, \tag{3.25}$$

$$q^{-\frac{1}{40}}\chi_{1,4}^{3,5}(q) = \sum_{\substack{l=1\\ \text{odd}}}^{\infty} \frac{q^{(l^2+2l)/4}}{(q)_l} = q^{\frac{3}{4}} \frac{\{1\}_{10}^+\{9\}_{10}^+}{\{2\}_{10}^-\{3\}_{10}^-\{5\}_{10}^-\{7\}_{10}^-\{8\}_{10}^-}. \tag{3.26}$$

Furthermore, we can apply (2.26) to combinations of the l.h.s. which yields

$$q^{-\frac{1}{40}}\left(\chi_{1,1}^{3,5}(q) \pm \chi_{1,4}^{3,5}(q)\right) = \sum_{l=0}^{\infty} \frac{(\pm)^l\, q^{(l^2+2l)/4}}{(q)_l} = \frac{\{3/4\}_{5/2}^{\pm}\{7/4\}_{5/2}^{\pm}}{\{2\}_5^-\{3\}_5^-\{5/2\}_{5/2}^+}. \tag{3.27}$$

We present a set of further Rogers–Ramanujan type identities derived in a similar way in Appendix E. The product sides of these identities are not unique in the sense that one may use the techniques discussed in Subsect. 2.4 and bring them, if possible, to the form (2.33), (2.34) or (2.35) (compare (3.27) and the corresponding formula in Appendix A). It is also worth noticing that, combining these identities further, we again obtain identities of the Rogers–Ramanujan type. For instance, multiplying (3.25) and (3.26), we find

$$\sum_{l_1,l_2=0}^{\infty} \frac{q^{l_1^2+l_2^2+l_1+2l_2}}{(q)_{2l_1}(q)_{2l_2+1}} = \frac{\{1\}_5^+\{4\}_5^+\left(\{5\}_5^+\right)^2}{\left(\{2\}_5^-\{3\}_5^-\right)^2}. \tag{3.28}$$

It should be mentioned that there exists a more general type of formulae than (3.24) (involving a q-deformed binomial factor) [20] which covers the whole range of characters in all minimal models. Therefore when our factorization technique applies we also have Rogers–Ramanujan identities for these more general types.

3.3. Quasi-particle representation. Once a character admits a factorizable form, it is easy to obtain a quasi-particle spectrum following the prescription of [21,22,6]. Let $\mathcal{P}(n,m)$ be the number of partitions of a positive integer n into m distinct non-negative integers and $\mathcal{Q}(n,m)$ be the number of partitions of a positive integer n into positive integers smaller or equal to m. In the theory of numbers the following formulae are well-known (e.g. [23]):

$$\sum_{n=0}^{\infty} \mathcal{P}(n,m)\, q^n = \frac{q^{m(m-1)/2}}{(q)_m}\,, \qquad \sum_{n=0}^{\infty} \mathcal{Q}(n,m)\, q^n = \frac{1}{(q)_m}\,. \tag{3.29}$$

Combining them with (1.12) and (1.13), we obtain

$$\{x\}_y^+ = \sum_{n,m=0}^{\infty} \mathcal{P}(n,m)\, q^{ny+mx} = \sum_{n,m=0}^{\infty} \mathcal{Q}(n,m)\, q^{(n+m(m-1)/2)y+mx}, \tag{3.30}$$

$$\frac{1}{\{x\}_y^-} = \sum_{n,m=0}^{\infty} \mathcal{Q}(n,m)\, q^{ny+mx} = \sum_{n,m=0}^{\infty} \mathcal{P}(n,m)\, q^{(n-m(m-1)/2)y+mx}. \tag{3.31}$$

We assume now the character to be of the form (3.22), and proceed in the usual way in order to derive the quasi-particle states. For this one interprets the characters as a partition function with $\chi(q = e^{-2\pi v/LkT}) = \sum_{l=0}^{\infty} P(E_l) e^{-E_l/kT}$, k being Boltzmann's constant, T the temperature, L the size of the quantizing system, v the speed of sound, E_l the energy of a particular level and $P(E_l)$ its degeneracy. The contribution of a single particle of type a and momentum $p_a^{i_a}$ (i_a being an additional internal quantum number) to the energy is assumed to be of the form $E_l = v \sum_{a=1}^{N+M} \sum_{i_a=1}^{l_a} \left| p_a^{i_a} \right|$. One has now the option to construct either a purely fermionic (in units of $2\pi/L$)

$$p_a^i(\boldsymbol{l}) = B_a + \frac{y}{2}\left(1 - \sum_{k=M+1}^{N+M} l_k\right) + yN_a^i \tag{3.32}$$

or purely bosonic spectrum (in units of $2\pi/L$)

$$p_a^i(\boldsymbol{l}) = B_a + \frac{y}{2}\left(1 - \sum_{k=1}^{M} l_k\right) + yM_a^i. \tag{3.33}$$

Table 1. Bosonic and fermionic spectrum for $\chi^{3,4}_{1,2}(q) = \frac{q^{1/24}}{\{1\}_2^-}$. k denotes the level and μ_k its degeneracy

k	μ_k	$p^i = 1 + 2M_i$	$p^i(l) = (2-l) + 2N_i$
1	1	$\|1\rangle$	$\|1\rangle$
2	1	$\|1,1\rangle$	$\|0,2\rangle$
3	2	$\|1,1,1\rangle$, $\|3\rangle$	$\|-1,1,3\rangle$, $\|3\rangle$
4	2	$\|1,1,1,1\rangle$, $\|1,3\rangle$	$\|-2,0,2,4\rangle$, $\|0,4\rangle$
5	3	$\|1,1,1,1,1\rangle$, $\|1,1,3\rangle$, $\|5\rangle$	$\|-3,-1,1,3,5\rangle$, $\|-1,1,5\rangle$, $\|5\rangle$
6	4	$\|1,1,1,1,1\rangle$, $\|3,3\rangle$, $\|1,5\rangle$, $\|1,1,1,3\rangle$	$\|-2,0,2,6\rangle$, $\|0,6\rangle$, $\|-4,-2,0,2,4,6\rangle$, $\|2,4\rangle$
7	5	$\|1,1,1,1,1,1\rangle$, $\|1,3,3\rangle$, $\|1,1,5\rangle$, $\|1,1,1,1,3\rangle$, $\|7\rangle$	$\|-5,-3,-1,1,3,5,7\rangle$, $\|-1,1,7\rangle$, $\|-1,3,5\rangle$, $\|-3,-1,1,3,7\rangle$, $\|7\rangle$

Here N_a^i are distinct positive integers and M_a^i are some arbitrary integers. The fermionic nature of this spectrum is here expressed through the fact that the integers N_a^i are all distinct, such that we have a Pauli principle. An example for such spectra is presented in Table 1. A particular interesting spectrum arises when we allow bosons and fermions

$$p_a^i = B_a + yN_a^i, \qquad p_b^i = B_b + yM_b^i \tag{3.34}$$

with $a \in \{1, M\}$ and $b \in \{M+1, N+M\}$. Notice now the dependence on $\boldsymbol{l}$ has vanished. When $N = M$ this may be interpreted in a supersymmetric way.

Following the procedure of this subsection, the answer to the question [24]: “How many fermionic representations are there for the characters of each model $\mathcal{M}(s,t)$?” would be *infinite* for factorizable characters due to the second relation in (1.14). One could also change the approach and start with a given spectrum and search for the related character [25] which shifts the problem to finding all possible integrable lattice models. A possible selection mechanism is given by using information from the massive models which in the conformal limit lead to certain models $\mathcal{M}(s,t)$. In this spirit for instance the choice A_1 and E_8 for the algebras of the related Cartan matrices in (3.24) appears quite natural.

3.4. Super-conformal characters. Linear combinations of characters may be found in various contexts as for instance when considering superconformal theories. The two N=1 unitary minimal superconformal extension of the Virasoro algebra are characterized by an integer l and a label $s = R, NS$, which refers to the Ramond or Neveu-Schwarz sector. The Virasoro central charge was found [26] to be

$$c(l) = \frac{3}{2}\left(1 - \frac{8}{l(l+2)}\right), \qquad l = 3, 4, \ldots . \tag{3.35}$$

The corresponding irreducible representations are highest weight representations with weights

$$H_{n,m}^{l,s} = \frac{((l+2)n - ml)^2 - 4}{8l(l+2)} + \frac{1}{16}\delta_{s,R} \quad , \tag{3.36}$$

where the labels are restricted as $1 \leq n \leq l-1$, $1 \leq m \leq l+1$ together with $n-m =$ even, odd when $s = NS$, $s = R$, respectively. Realizing these models as $S\hat{U}(2)_{l-2} \otimes S\hat{U}(2)_2/S\hat{U}(2)_{l+2}$-cosets the corresponding characters $\Xi^{l,s}_{n,m}(q)$ were constructed in [10]. One notices from (1.4) and (3.35) that $c(3) = c(4,5)$ and indeed, applying twice the GKO-sumrules one may identify supersymmetric characters with linear combinations of some non-supersymmetric Virasoro characters

$$\Xi^{3,NS}_{1,1}(q) = \chi^{4,5}_{1,1}(q) + \chi^{4,5}_{1,4}(q), \qquad \Xi^{3,\tilde{NS}}_{1,1}(q) = \chi^{4,5}_{1,1}(q) - \chi^{4,5}_{1,4}(q), \tag{3.37}$$

$$\Xi^{3,NS}_{1,3}(q) = \chi^{4,5}_{1,2}(q) + \chi^{4,5}_{1,3}(q), \qquad \Xi^{3,\tilde{NS}}_{1,3}(q), = \chi^{4,5}_{1,2}(q) - \chi^{4,5}_{1,3}(q), \tag{3.38}$$

$$\Xi^{3,R}_{2,1}(q), = \chi^{4,5}_{2,1}(q), \qquad \Xi^{3,R}_{2,3}(q), = \chi^{4,5}_{2,2}(q). \tag{3.39}$$

Notice that all these characters factorize (see Appendix A for the explicit formulae). Moreover, they admit the form (2.35) (which is due to Rocha-Caridi [1]) as well as the form (2.33) (see Appendix A). It is interesting that the latter does not appear to be manifestly supersymmetric. We observe easily the property for these expressions under the T-modular transformation (assuming y to be an integer, the effect of this transformation is that $\{x\}^{\pm}_y \rightarrow \{x\}^{\pm}_y$ when x is an integer and $\{x\}^{\pm}_y \rightarrow \{x\}^{\mp}_y$ when x is a semi-integer) which relates $\Xi^{l,NS}_{n,m}(q)$ and $\Xi^{l,\tilde{NS}}_{n,m}(q)$ and leaves $\Xi^{l,R}_{n,m}(q)$ invariant. Fermionic representations for all characters $\Xi^{l,s}_{n,m}(q)$ were found in [27] and we leave it for future investigations to settle the question whether they also factorize or not. As in the non-supersymmetric case the modular properties of these characters [28] will certainly turn out to be useful.

3.5. Modular invariant partition functions. Modular invariant partition functions for minimal models are given by (up to an overall coefficient)

$$Z^{s,t}(q,\bar{q}) = \sum_{n,n',m,m'} Z^{m,m'}_{n,n'} \, \chi^{s,t}_{n,m}(q) \, \overline{\chi^{s,t}_{n',m'}(q)}. \tag{3.40}$$

For the so-called main sequence (in the terminology of [17]), or (A_{s-1}, A_{t-1}) type, we have $Z^{m,m'}_{n,n'} = \delta_{n,n'}\delta_{m,m'}$. Bearing in mind factorizability of all characters in the $\mathcal{M}(2,t)$ and $\mathcal{M}(3,t)$ models, one can rewrite the corresponding partition functions as a sum of products of the type (2.33). This allows, in particular, to apply the technique of Subsect. 3.3 and obtain quasi-particle representations for these partition functions.

Besides the main sequence some minimal models possess other modular invariants (complementary sequences) [3,17,29] of the type (3.40) with more general $Z^{m,m'}_{n,n'}$. In particular, for $\mathcal{M}(4k,t)$ and $\mathcal{M}(4k+2,t)$ ((D_{2k+1}, A_{t-1}) and (D_{2k+2}, A_{t-1}) type, respectively) the non-diagonal part of $Z^{m,m'}_{n,n'}$ is $z_{nm}\delta_{n,n'}\delta_{m,t-m'}$. In this case (3.40) involves not only squares of modules of single characters but also those of sums of characters of the type (2.11). For $t = 3$ all of these sums are factorizable and we can represent the corresponding partition functions as a sum of products (of the type (2.35) in general). Thus, for such partition functions we also can obtain quasi-particle representations.

3.6. Partition functions in boundary CFT. A partition function of a conformal theory on a manifold with boundaries, say on a cylinder, is expressed as a sum of characters of a single copy of the Virasoro algebra [30]

$$Z_{\alpha,\beta}(q) = \sum_h N^h_{\alpha\beta}\chi_h(q), \tag{3.41}$$

where (α, β) is a pair of boundary conditions, $\chi_h(q)$ denotes a character of given weight h, and $N^h_{\alpha\beta}$ are multiplicities (expressible in terms of (2.17) and also related to the fusion rules).

It is interesting that in some cases $Z_{\alpha,\beta}(q)$ is just a factorizable sum (or several such sums) of type (2.11), so we can rewrite it in the product form. For instance, for the critical 3-state Potts model (corresponding to $\mathcal{M}(5, 6)$) there are three microscopic states A, B and C, and for some of possible partition functions we find

$$Z_{A,F}(q) = \chi^{5,6}_{1,2}(q) + \chi^{5,6}_{4,2}(q) = q^{\frac{11}{120}} \frac{1}{\{1\}^-_{5/2}\{3/2\}^-_{5/2}}, \tag{3.42}$$

$$Z_{BC,F}(q) = \chi^{5,6}_{2,2}(q) + \chi^{5,6}_{3,2}(q) = q^{-\frac{1}{120}} \frac{1}{\{1/2\}^-_{5/2}\{2\}^-_{5/2}}, \tag{3.43}$$

where F stands for the free boundary condition. As we mentioned in Subsect. 2.3, such an expression may be interpreted as a character of a module generated by bosonic operators (in fact, (3.17) shows that (3.42) and (3.43) coincide with the characters of $\mathcal{M}(2, 5)$ of an argument $q^{1/2}$). Also, this form of a partition function allows for a direct extraction of a quasi-particle spectrum which, (in the spirit of Subsect. 3.3) in particular, can be used to study connections between theories with distinct boundary conditions.

Conclusion

We have shown how to obtain the factorized form of a single Virasoro character on the base of the Gauß-Jacobi and Watson identities by exploiting the quasi-classical asymptotics of the usual sum representation. We have applied this method also to the factorization of a linear combination of two Virasoro characters and found the explicit formulae (2.23), (2.26) and (2.27). We presented a rigorous proof that besides the obtained expressions no other differences of two Virasoro characters of the type (2.11) are factorizable in the form (2.33). It is a remarkable fact, which certainly needs some deeper understanding, that just like for the single characters none of the Macdonald identities, other than the ones corresponding to the $A_1^{(1)}$ and $A_2^{(2)}$ algebras need to be invoked. We employed the obtained factorized versions of the characters in order to derive a set of new identities, e.g. (3.7)–(3.10), in a very economical way. Some particular cases of these identities coincide with formulae derived originally in [19], however now the proof has simplified considerably. As was already pointed out in [19], these identities belong to a class which is closely related, but not derivable, from a repeated use of the GKO-sumrules [10]. It is therefore suggestive to assume that the new identities are related to some higher sumrules. A systematic classification of identities obtainable from factorised combinations of Virasoro characters will be presented elsewhere. It is also conceivable, that the presented method will be applicable to non-minimal models like parafermionic models, i.e. $\hat{SU}(2)_k/\hat{U}(1)_k$-coset, or general N=1,2,4 supersymmetric models. Concerning the quasi-particle representation of the Virasoro characters with their relation to lattice models, the factorized versions constitute a suitable starting point for a more detailed analysis, as for instance in [22].

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Appendix A

Here we will present some examples of the inverse product representation for characters and linear combinations of characters in some unitary and non-unitary models. For shortness we omit the argument q on l.h.s. and use the notation $\{x_1; \ldots ; x_n\}_y^{\pm} := \{x_1\}_y^{\pm} \ldots \{x_n\}_y^{\pm}$.

$$\chi_{1,1}^{3,4} \pm \chi_{1,3}^{3,4} = q^{-\frac{1}{48}} \frac{1}{\{1/2\}_1^{\mp}\{1\}_1^{+}}, \qquad \chi_{1,2}^{3,4} = q^{\frac{1}{24}} \frac{1}{\{1\}_2^{-}},$$

$$\chi_{2,1}^{4,5} = q^{\frac{49}{120}} \frac{1}{\{1;4\}_5^{-}\{3;5;7\}_{10}^{-}}, \qquad \chi_{2,2}^{4,5} = q^{\frac{1}{120}} \frac{1}{\{2;3\}_5^{-}\{1;5;9\}_{10}^{-}},$$

$$\chi_{1,1}^{4,5} \pm \chi_{1,4}^{4,5} = q^{-\frac{7}{240}} \frac{1}{\{3/2;5/2;7/2\}_5^{\mp}\{5\}_5^{+}\{2;8\}_{10}^{-}},$$

$$\chi_{1,2}^{4,5} \pm \chi_{1,3}^{4,5} = q^{\frac{17}{240}} \frac{1}{\{1/2;5/2;9/2\}_5^{\mp}\{5\}_5^{+}\{4;6\}_{10}^{-}},$$

$$\chi_{1,1}^{5,6} - \chi_{1,5}^{5,6} = q^{-\frac{1}{30}} \frac{1}{\{2;8\}_{10}^{-}}, \qquad \chi_{2,1}^{5,6} - \chi_{2,5}^{5,6} = q^{\frac{11}{30}} \frac{1}{\{4;6\}_{10}^{-}},$$

$$\chi_{1,2}^{5,6} \pm \chi_{1,4}^{5,6} = \frac{q^{\frac{11}{120}}}{\{1;4\}_5^{-}\{3/2;7/2\}_5^{\mp}}, \qquad \chi_{2,2}^{5,6} \pm \chi_{2,4}^{5,6} = \frac{q^{-\frac{1}{120}}}{\{2;3\}_5^{-}\{1/2;9/2\}_5^{\mp}},$$

$$\chi_{1,1}^{6,7} - \chi_{1,6}^{6,7} = \frac{q^{-\frac{1}{28}}}{\{3;4\}_7^{-}\{2;12\}_{14}^{-}}, \qquad \chi_{1,2}^{6,7} - \chi_{1,5}^{6,7} = \frac{q^{\frac{3}{28}}}{\{1;6\}_7^{-}\{4;10\}_{14}^{-}},$$

$$\chi_{1,3}^{6,7} - \chi_{1,4}^{6,7} = \frac{q^{\frac{19}{28}}}{\{2;5\}_7^{-}\{6;8\}_{14}^{-}},$$

$$\chi_{2,1}^{6,7} \pm \chi_{2,6}^{6,7} = \frac{q^{\frac{19}{56}}}{\{1;3;4;6\}_7^{-}\{5/2;9/2\}_7^{\mp}},$$

$$\chi_{2,2}^{6,7} \pm \chi_{2,4}^{6,7} = \frac{q^{-\frac{1}{56}}}{\{1;2;5;6\}_7^{-}\{3/2;11/2\}_7^{\mp}},$$

$$\chi_{2,3}^{6,7} \pm \chi_{2,4}^{6,7} = q^{\frac{3}{56}} \frac{1}{\{2;3;4;5\}_7^{-}\{1/2;13/2\}_7^{\mp}},$$

$$\chi_{1,1}^{2,5} = q^{\frac{11}{60}} \frac{1}{\{2;3\}_5^{-}}, \qquad \chi_{1,2}^{2,5} = q^{-\frac{1}{60}} \frac{1}{\{1;4\}_5^{-}},$$

$$\chi_{1,1}^{3,5} \pm \chi_{1,4}^{3,5} = q^{\frac{1}{40}} \frac{1}{\{2;8\}_{10}^{-}\{3/4;7/4\}_{\frac{5}{2}}^{\mp}\{3/2;5/2;7/2\}_{5}^{+}},$$
$$\chi_{1,2}^{3,5} \pm \chi_{1,3}^{3,5} = q^{-\frac{1}{40}} \frac{1}{\{4;6\}_{10}^{-}\{1/4;9/4\}_{\frac{5}{2}}^{\mp}\{1/2;5/2;9/2\}_{5}^{+}}.$$

Appendix B

In this appendix we present a sample proof for the identities of the type (2.22)–(2.23) and (2.26)–(2.27), that is for the factorization of the sum or difference of two Virasoro characters related to minimal models. The proof is based on a systematic exploitation of the Gauß-Jacobi and Watson identities (1.20)–(1.23). We have to compare the l.h.s. of these expressions with the sum or difference of characters given by (1.7),

$$\chi_{n,m}^{s,t}(q) \pm \chi_{n,t-m}^{s,t}(q) = \frac{q^{h_{n,m}^{s,t} - \frac{c(s,t)}{24}}}{(q)_\infty} \sum_{k=-\infty}^{\infty} q^{stk^2} \Big(q^{k(nt-ms)} - q^{k(nt+ms)+nm} \pm q^{k(nt+ms-st)+\Delta h_{n,m}^{s,t}} \mp q^{k(nt-ms+st)+n(t-m)+\Delta h_{n,m}^{s,t}} \Big). \quad \text{(B.1)}$$

Here the quantity $\Delta h_{n,m}^{s,t}$ is defined by (2.12) and we assume $n < s/2$, $m < t/2$, so that $\Delta h_{n,m}^{s,t} > 0$. We outline the proof for the identity (2.27). All other proofs work along the same lines.

Recall that (2.27) has been conjectured to be a particular case of (1.21) for $a = \Delta h_{n,m}^{s,t}$ and $b = nm$ provided that the condition $s = 6n$ holds. Notice that substitution of the latter relation into (2.12) yields $a = nt - 2nm$. In order to produce the right number of terms for a possible comparison with (B.1), we have to split the sum in the l.h.s. of (1.21) into two new sums – over even and odd k. Then the l.h.s. of (1.21) acquires the form

$$\sum_{k=-\infty}^{\infty} q^{k^2(6a+12b)} \Big(q^{k(a-4b)} + q^{k(7a+8b)+2a+b} - q^{k(a+8b)+b} - q^{k(7a+20b)+2a+8b} \Big),$$

which, upon substitution of the explicit values for a and b and the relation $s = 6n$, becomes

$$\sum_{k=-\infty}^{\infty} q^{stk^2} \Big(q^{k(nt-ms)} + q^{k(nt-ms+st)+2nt-3nm} - q^{k(nt+ms)+nm} - q^{k(nt+ms+st)+2nt+4nm} \Big).$$

We see that the first, second and third terms here exactly match the first, fourth and second terms on the r.h.s. in (B.1), respectively. Making the shift $k \to k-1$ in the last term, we achieve that it coincides with the third term in (B.1). This completes the proof.

Appendix C

Here we will prove the following statement: the factorization of the difference of two minimal Virasoro characters in the form

$$\chi^{s,t}_{n,m}(q) - \chi^{s,t}_{s-n,m}(q) = \frac{q^{h^{s,t}_{n,m} - \frac{c(s,t)}{24}}}{(q)_\infty}\left(\hat{\chi}^{s,t}_{n,m}(q) - q^{\Delta h}\hat{\chi}^{s,t}_{s-n,m}(q)\right) = \frac{q^{h^{s,t}_{n,m} - \frac{c(s,t)}{24}}}{\prod_i^N \{x_i\}^-_b}, \tag{C.1}$$

where $0 < x_1 < \ldots < x_N \leq b$, is up to the symmetries (1.8) only possible for $s = 3n, 4n, 6n$. Here Δh stands for $\Delta h^{s,t}_{n,m}$ defined in (2.12), and we assume $n < s/2$, $m < t/2$, so that $\Delta h^{s,t}_{n,m} > 0$.

Our argumentation goes along the lines of the proof for the factorization of single characters given in [2]. Surprisingly it is enough to investigate the first five terms in the sum, which for the incomplete character may be identified uniquely

$$\hat{\chi}^{s,t}_{n,m}(q) = 1 - q^{nm} - q^{(s-n)(t-m)} + q^{ts+sm-tn} + q^{ts+tn-sm} + \ldots . \tag{C.2}$$

For the difference of the two characters they read

$$\hat{\chi}^{s,t}_{n,m}(q) - q^{\Delta h}\hat{\chi}^{s,t}_{s-n,m}(q) = 1 - q^{nm} - q^{\Delta h} + q^{\Delta h + m(s-n)} + q^{\Delta h + n(t-m)} + \ldots . \tag{C.3}$$

For definiteness we choose $sm < nt$ (so that $\Delta h + m(s-n) < \Delta h + n(t-m)$), since the other case may be obtained from the symmetry properties. The negative terms in (C.3) allow us to write down the first two factors in the product

$$\hat{\chi}^{s,t}_{n,m}(q) - q^{\Delta h}\hat{\chi}^{s,t}_{s-n,m}(q) = (1 - q^{nm})(1 - q^{\Delta h})\ldots , \tag{C.4}$$

which means that after expanding we will generate a term $q^{nm+\Delta h}$. Since $nm + \Delta h < \Delta h + m(s-n)$, we have to include a factor $(1 - q^{nm+\Delta h})$ on the r. h. s. of (C.4) in order to cancel this term. Expanding once more we will generate new terms, which in turn have to be cancelled by additional factors on the right hand side of (C.4) until we obtain the matching condition $\alpha nm + \beta \Delta h = \Delta h + m(s-n)$ with positive integers α and β. At first sight it seems a formidable task to bring some systematics into this analysis. However, it was observed in [2] that this procedure will terminate when $\alpha + \beta = 5$. Actually also one case from level 6 might be possible.

Performing this analysis up to that level one obtains

$$\begin{aligned} &1 - q^{nm} - q^{\Delta h} + q^{\alpha nm + \beta \Delta h} + \ldots \\ &\quad = (1 - q^{nm})(1 - q^{\Delta h})(1 - q^{nm+\Delta h})(1 - q^{2nm+\Delta h}) \\ &\quad\quad \times (1 - q^{nm+2\Delta h})(1 - q^{3nm+\Delta h})(1 - q^{nm+3\Delta h})(1 - q^{2nm+2\Delta h}) \\ &\quad\quad \times (1 - q^{4nm+\Delta h})(1 - q^{nm+4\Delta h})(1 - q^{3nm+2\Delta h})^2(1 - q^{2nm+3\Delta h})^2 \ldots , \end{aligned}$$

where in this expression $\alpha + \beta > 5$. It is the occurrence of the quadratic terms $(1 - q^{3nm+2\Delta h})^2$ and $(1 - q^{2nm+3\Delta h})^2$ which allows us to stop at this point, since they may never be cancelled against factors within $(q)_\infty$ and we can therefore restrict the investigation to the cases $2 \leq \alpha + \beta \leq 5$. Commencing with the case $\alpha + \beta = 5$ we obtain two matching conditions, that is for the two smallest powers of the positive terms

$$\left.\begin{aligned} \tfrac{st}{4} + \tfrac{ms}{2} - \tfrac{nt}{2} &= 3nm + 2\Delta h \\ \tfrac{st}{4} - \tfrac{ms}{2} + \tfrac{nt}{2} &= 2nm + 3\Delta h \end{aligned}\right\} \quad \Rightarrow s = 6n - 2m\frac{s}{t}.$$

Since n is positive, m is strictly smaller than t, $\langle s,t\rangle = 1$ and $t = 2m$ produces zero on the left hand side of (C.1), the case $\alpha + \beta = 5$ will never produce any solution. We may also encounter the situation

$$\left.\begin{array}{l} \frac{st}{4} + \frac{ms}{2} - \frac{nt}{2} = 4nm + \Delta h \\ \frac{st}{4} - \frac{ms}{2} + \frac{nt}{2} = 2nm + 2\Delta h \end{array}\right\} \Rightarrow s = 5n \;\text{ and }\; t = 6m.$$

In the remaining possibilities we only obtain one matching condition, that is for p. The case $\alpha = \beta = 2$ leads to the condition $2nm + 2\Delta h = sm - nm + \Delta h$ which amounts to

$$m = t\frac{(s-2n)}{(6s-16n)}.$$

However, substitution of this relation into the condition $sm < nt$ leads to $s(s-2n) < n(6s-16n)$, or equivalently, $(s-4n)^2 < 0$, which is impossible.

The other cases yield

$$\begin{aligned} nm + \Delta h = sm - nm + \Delta h \quad &\Rightarrow \quad s = 2n, \\ nm + 2\Delta h = sm - nm + \Delta h \quad &\Rightarrow \quad s = 2n \;\text{ or }\; t = 6m, \\ 2nm + \Delta h = sm - nm + \Delta h \quad &\Rightarrow \quad s = 3n, \\ nm + 3\Delta h = sm - nm + \Delta h \quad &\Rightarrow \quad s = 2n \;\text{ or }\; t = 4m, \\ 3nm + \Delta h = sm - nm + \Delta h \quad &\Rightarrow \quad s = 4n \quad , \\ 5nm + \Delta h = sm - nm + \Delta h \quad &\Rightarrow \quad s = 6n \quad . \end{aligned}$$

We observe that we recover the cases we claimed to factorize in the form (C.4), which concludes the proof.

Appendix D

We will now provide a sample proof for the identities (3.7)–(3.10). For $n = m = 1$ some very involved proof which employs identities of theta functions may be found in [19]. With the help of the product representations (2.22)–(2.27) such identities may be derived without any effort. We demonstrate this just for Eqs. (3.9) with the upper sign, the remaining equations may be derived in a similar way. First of all we notice that

$$\begin{aligned} h^{3n,2m}_{n,m} - \frac{c(3n,2m)}{24} + h^{4n,5m}_{n,m} - \frac{c(4n,5m)}{24} = & h^{6n,5m}_{2n,2m} - \frac{c(6n,5m)}{24} \\ & + h^{3n,4m}_{n,m} - \frac{c(3n,4m)}{24}. \end{aligned} \tag{D.1}$$

After cancelling $(q)^2_\infty$ on both sides of (3.9) for the upper sign we obtain for the left hand side upon using (1.14) (we omit here the labels nm in order to avoid lengthy formulae and imagine just for now that $\{x\}^-_y$ should always be understood as $\{xnm\}^-_{ynm}$),

$$\begin{aligned} &\hat{\chi}^{3n,2m}_{n,m}\left(\hat{\chi}^{4n,5m}_{n,m} + \hat{\chi}^{4n,5m}_{n,4m}\right) \\ &= \left(\{1\}^-_1\right)\left(\{1\}^-_5\{4\}^-_5\{5\}^-_5\left\{\frac{3}{2}\right\}^+_5\left\{\frac{5}{2}\right\}^+_5\left\{\frac{7}{2}\right\}^+_5\right) \end{aligned}$$

$$= \{1\}_4^- \{3\}_4^- \{2\}_2^- \{1\}_{10}^- \{6\}_{10}^- \{4\}_{10}^- \{9\}_{10}^- \{5\}_5^- \left\{\frac{3}{2}\right\}_5^+ \left\{\frac{5}{2}\right\}_5^+ \left\{\frac{7}{2}\right\}_5^+$$
$$= \left(\{1\}_4^- \{3\}_4^- \{2\}_2^- \left\{\frac{1}{2}\right\}_2^+ \left\{\frac{3}{2}\right\}_2^+\right) \left(\{4\}_{10}^- \{6\}_{10}^- \{5\}_5^- \left\{\frac{1}{2}\right\}_5^- \left\{\frac{9}{2}\right\}_5^-\right)$$
$$= \left(\hat{\chi}_{n,m}^{3n,4m} + \hat{\chi}_{n,3m}^{3n,4m}\right) \left(\hat{\chi}_{2n,2m}^{6n,5m} - \hat{\chi}_{2n,3m}^{6n,5m}\right).$$

Here we have used several times the identities (1.14).

Appendix E

We complement the list started in Subsect. 3.2 of the Rogers–Ramanujan type identities obtained by combining our product formulae for (combinations of) characters with the results of [21]. We adopt the notations explained in Appendix A:

$$q^{1/48} \chi_{1,1}^{3,4} = \sum_{\substack{l=0 \\ \text{even}}}^{\infty} \frac{q^{l^2/2}}{(q)_l} = \frac{1}{\{2;\,14\}_{16}^- \{3;\,4;\,5\}_8^-}, \tag{E.1}$$

$$q^{1/48} \chi_{1,3}^{3,4} = \sum_{\substack{l=1 \\ \text{odd}}}^{\infty} \frac{q^{l^2/2}}{(q)_l} = q^{1/2} \frac{1}{\{6;\,10\}_{16}^- \{1;\,4;\,7\}_8^-}, \tag{E.2}$$

$$q^{1/48} \left(\chi_{1,1}^{3,4} \pm \chi_{1,3}^{3,4}\right) = \sum_{l=0}^{\infty} \frac{(\pm)^l\, q^{l^2/2}}{(q)_l} = \{1/2\}_1^{\pm}, \tag{E.3}$$

$$q^{-1/24} \chi_{1,2}^{3,4} = \sum_{\substack{l=0 \\ \text{even}}}^{\infty} \frac{q^{(l^2-l)/2}}{(q)_l} = \sum_{\substack{l=1 \\ \text{odd}}}^{\infty} \frac{q^{(l^2-l)/2}}{(q)_l} = \frac{1}{\{1\}_2^-}, \tag{E.4}$$

$$q^{1/30} \chi_{1,3}^{5,6} = \sum_{\substack{l_1,l_2=0 \\ l_1+2l_2=\pm 1 (\mathrm{mod}\, 3)}}^{\infty} \frac{q^{2(l_1^2+l_1^2+l_1 l_2)/3}}{(q)_{l_1} (q)_{l_2}} = \frac{q^{2/3}}{\{1;\,2\}_3^- \{6;\,9\}_{15}^-}, \tag{E.5}$$

$$q^{\frac{1}{40}} \chi_{1,2}^{3,5} = \sum_{\substack{l=0 \\ \text{even}}}^{\infty} \frac{q^{l^2/4}}{(q)_l} = \frac{\{3;\,7\}_{10}^+}{\{1;\,4;\,5;\,6;\,9\}_{10}^-}, \tag{E.6}$$

$$q^{\frac{1}{40}} \chi_{1,3}^{3,5} = \sum_{\substack{l=1 \\ \text{odd}}}^{\infty} \frac{q^{l^2/4}}{(q)_l} = q^{1/4} \frac{\{2;\,8\}_{10}^+}{\{1;\,4;\,5;\,6;\,9\}_{10}^-}, \tag{E.7}$$

$$q^{\frac{1}{40}} \left(\chi_{1,2}^{3,5} \pm \chi_{1,3}^{3,5}\right) = \sum_{l=0}^{\infty} \frac{(\pm)^l\, q^{l^2/4}}{(q)_l} = \frac{\{1/4\}_{5/2}^{\pm} \{9/4\}_{5/2}^{\pm}}{\{1;\,4\}_5^- \{5/2\}_{5/2}^+}. \tag{E.8}$$

More identities will be given elsewhere [31].

References

1. Rocha-Caridi, A.: Vertex Operators in Mathematics and Physics. ed. J. Lepowsky et al, Berlin: Springer, 1985
2. Christe, P.: Int. J. Mod. Phys. **29**, 5271 (1991)
3. Cappelli, A., Itzykson, C., and Zuber, J.B.: Nucl. Phys. **B280**, 445 (1987); Commun. Math. Phys. **113**, 1 (1987)
4. Kellendonk, J., Rösgen, M., and Varnhagen, R.: Int. J. Mod. Phys. **A9**, 1009 (1994)
5. Eholzer, W., and Skoruppa, N.-P.: Phys. Lett. **B388**, 82 (1996)
6. Bytsko, A.G., and Fring, A.: Nucl. Phys. **B521**, 573 (1998)
7. Rogers, L.J.: Proc. London Math. Soc. **25**, 318 (1894);
   Ramanujan, S.: J. Indian Math. Soc. **6**, 199 (1914);
   Schur, I.: Berliner Sitzungsberichte **23**, 301 (1917)
8. Belavin, A.A., Polyakov, A.M., and Zamolodchikov, A.B.: Nucl. Phys. **B241**, 333 (1984)
9. Friedan, D., Qiu, Z., and Shenker, S.: Phys. Rev. Lett. **52**, 1575 (1984); Commun. Math. Phys. **107**, 535 (1986)
10. Goddard, P., Kent, A., and Olive, D.: Commun. Math. Phys. **103**, 105 (1986)
11. Feigin, B.L., and Fuchs, D.B.: Funct. Anal. Appl. **17**, 241 (1983)
12. Kac, V.G.: Infinite dimensional Lie algebras. Cambridge: Cambridge U. Press, 1990
13. Watson, G.N.: J. London Math. Soc. **4**, 39 (1929)
14. Faddeev, L.D., and Kashaev, R.M.: Mod. Phys. Lett. **A9**, 427 (1994)
15. Lewin, L.: Polylogarithms and associated functions. Amsterdam: North-Holland, 1981
16. Blöte, H., Nichtingale, M.P., and Cardy, J.L.: Phys. Rev. Lett. **56**, 742 (1986);
    Affleck, I.: Phys. Rev. Lett. **56**, 746 (1986);
    Itzykson, C., Saleur, H., and Zuber, J.B.: Europhys. Lett. **2**, 91 (1986)
17. Itzykson, C., and Zuber, J.B.: Nucl. Phys. **B275**, 580 (1986)
18. Macdonald, I.G.: Invent. Math. **15**, 91 (1972)
19. Taormina, A.: Commun. Math. Phys. **165**, 69 (1994)
20. Berkovich, A., McCoy, B.M., and Schilling, A.: Commun. Math. Phys. **191**, 325 (1998)
21. Kedem, R., Klassen, T.R., McCoy, B.M., and Melzer, E.: Phys. Lett. **B304**, 263 (1993); **B307**, 68 (1993)
22. Belavin, A.A., and Fring, A.: Phys. Lett. **B409**, 199 (1997)
23. Hardy, G.H., and Wright, E.M.: An introduction to the theory of numbers. Oxford: Clarendon Press, 1965
24. Berkovich, A., and McCoy, B.M.: “Proceedings of the ICM 1998, Vol.III.” Berlin: DMV, 1998, p. 163
25. McCoy, B.M.: Private communication
26. Friedan, D., Qiu, Z., and Shenker, S.: Phys. Lett. **B151**, 37 (1985)
27. Baver, E., and Gepner, D.: Phys. Lett. **B372**, 231 (1996)
28. Capelli, A.: Phys. Lett. **B185**, 82 (1987);
    Minces, P., Namazie, M.A., and Nunez, C.: Phys. Lett. **B422**, 117 (1998)
29. Gepner, D.: Nucl. Phys. **B287**, 111 (1987);
    Kato, A.: Mod. Phys. Lett. **A2**, 585 (1987)
30. Cardy, J.L.: Nucl. Phys. **B324**, 581 (1989)
31. Bytsko, A.G.: J. Phys. **A32**, 8045 (1999)

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