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Form factors from free fermionic Fock fields, the Federbush model

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Abstract

By representing the field content as well as the particle creation operators in terms of fermionic Fock operators, we compute the corresponding matrix elements of the Federbush model. Only when these matrix elements satisfy the form factor consistency equations involving anyonic factors of local commutativity, the corresponding operators are local. We carry out the ultraviolet limit, analyse the momentum space cluster properties and demonstrate how the Federbush model can be obtained from the $SU(3)_3$-homogeneous sine-Gordon model. We propose a new class of Lagrangians which constitute a generalization of the Federbush model in a Lie algebraic fashion. We evaluate the associated scattering matrices from first principles, which can alternatively also be obtained in a certain limit of the homogeneous sine-Gordon models. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

The analysis of the structure and properties, as well as the evaluation of exact form factors, is one of the central problems in $(1+1)$-dimensional quantum field theories. One of the main reasons for their distinct role is that they serve to compute very efficiently correlations functions of local operators $\mathcal{O}(x)$. Instead of a perturbative expansion in the coupling constant one may expand the correlation functions in terms of exact expressions of n-particle form factors, that is the matrix element of a local operator $\mathcal{O}(x)$ located at the origin between a multiparticle in-state and the vacuum

$$F_n^{\mathcal{O}|\mu_1\cdots\mu_n}(\theta_1,\ldots,\theta_n) \equiv \left\langle \mathcal{O}(0) Z_{\mu_1}^\dagger(\theta_1)\cdots Z_{\mu_n}^\dagger(\theta_n)\right\rangle_{\text{in}}. \tag{1}$$

The operators $Z_\mu^\dagger(\theta)$ are creation operators for a particle of type μ as a function of the rapidity θ.

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Since the original proposal of this method to evaluate correlation functions [1], various schemes have been suggested to compute these objects. One of the original approaches is modeled in spirit closely on the set up for the determination of exact scattering matrices. It consists of solving a system of consistency equations which have to hold for the n-particle form factors based on some natural physical assumptions, like unitarity, crossing and bootstrap fusing properties [1,2]

$$F_n^{\mathcal{O}|\cdots\mu_i\mu_{i+1}\cdots}(\ldots,\theta_i,\theta_{i+1},\ldots) = F_n^{\mathcal{O}|\cdots\mu_{i+1}\mu_i\cdots}(\ldots,\theta_{i+1},\theta_i,\ldots)S_{\mu_i\mu_{i+1}}(\theta_{i,i+1}), \tag{2}$$

$$F_n^{\mathcal{O}|\mu_1\cdots\mu_n}(\theta_1+2\pi i,\ldots,\theta_n) = \gamma_{\mu_1}^{\mathcal{O}} F_n^{\mathcal{O}|\mu_2\cdots\mu_n\mu_1}(\theta_2,\ldots,\theta_n,\theta_1), \tag{3}$$

$$\begin{aligned}&\mathrm{Res}_{\bar\theta\to\theta_0} F_{n+2}^{\mathcal{O}|\bar\mu\mu\mu_1\cdots\mu_n}(\bar\theta+i\pi,\theta_0,\theta_1,\ldots,\theta_n)\\ &\quad = i\left(1-\gamma_\mu^{\mathcal{O}}\prod_{l=1}^{n} S_{\mu\mu_l}(\theta_{0l})\right) F_n^{\mathcal{O}|\mu_1\cdots\mu_n}(\theta_1,\ldots,\theta_n).\end{aligned} \tag{4}$$

Here $\gamma_\mu^{\mathcal{O}}$ is the factor of so-called local commutativity defined through the equal time exchange relation of the local operator $\mathcal{O}(x)$ and the field $\mathcal{O}_\mu(y)$ associated to the particle creation operators $Z_\mu^\dagger(\theta)$

$$\mathcal{O}_\mu(x)\mathcal{O}(y) = \gamma_\mu^{\mathcal{O}}\mathcal{O}(y)\mathcal{O}_\mu(x) \quad \text{for } x^1 > y^1. \tag{5}$$

The factor $\gamma_\mu^{\mathcal{O}}$ is very often omitted in the analysis or simply taken to be one, but it can be seen that already in the Ising model it is needed to set up the equations consistently [3]. A consequence of its presence is that a frequently made statement has to be revised, namely, that (2)–(4) constitute operator independent equations, which require as the only input the S-matrices $S_{ij}(\theta_{ij})$ between particles of type i and j as a function of the rapidity difference $\theta_{ij} \equiv \theta_i - \theta_j$. In the following manuscript we demonstrate that apart from ± 1, which already occur in the literature, this factor can be a nontrivial phase. Thus the form factor consistency equations contain also explicitly nontrivial properties of the operators.

Trying to find solutions to these equations has been pursuit successfully for many models and has led to the determination of closed exact expressions for n-particle form factors for a wide class of local operators $\mathcal{O}(x)$, e.g., [1,2].

Alternatively some authors develop methods which borrow ideas which have proven to be very powerful in the context of conformal field theory, where the use of symmetries and their related algebras has led to a successful determination of correlation functions [4]. Yet, the most direct way to compute the matrix elements in (1) is to find explicit representations for the operators $Z_\mu^\dagger(\theta)$ and $\mathcal{O}(x)$. For instance in the context of lattice models this is a rather familiar situation and one knows how to compute matrix elements of the type (1) directly. The problem is then reduced to a purely computational task (albeit nontrivial), which may, for instance, be solved by well-known techniques of algebraic Bethe ansatz type, e.g., [5]. In the context of field theory a similar way of attack to the problem has been followed by exploiting a free field representation for the operators $Z_\mu^\dagger(\theta)$ and $\mathcal{O}(x)$, in form of Heisenberg algebras or their q-deformed version. So far a successful computation of the

n-particle form factors with this approach is limited to a rather restricted set of models and in particular, for the sine-Gordon model, which is a model extensively studied by means of other approaches [2,6], only the free fermion point can be treated successfully [7,8] so far. One of the main purpose of this manuscript is to advocate yet another approach, namely, the evaluation of the matrix elements (1) based on an expansion of the operators in the conventional fermionic Fock space. Recalling the well-known fact that in $1+1$ space–time dimensions the notions of spin and statistics are not intrinsic, it is clear that both approaches are legitimate. Since the model we mainly consider in this manuscript, the Federbush model, is closely related to complex free fermions the usage of fermionic Fock operators seems natural. Nonetheless, we expect this procedure to hold in more generality and to allow an extension to other models.

Our manuscript is organized as follows: in Section 2 we recall an explicit fermionic free field representation for the particle creation operators $Z_{\mu}^{\dagger}(\theta)$ occurring in (1) valid for all diagonal scattering matrices. In Section 3 we treat the complex free fermion, we provide a generic expression for a potentially local operator and specialize it to particular operators whose form factors we directly compute, namely, the order and disorder field and various components of the energy–momentum tensor. In Section 4 we extend this analysis to the Federbush model and show in particular how it is related to homogeneous sine-Gordon (HSG) models on the level of the scattering matrix. In addition we analyze the momentum space cluster property. We pay special attention to the factor of local commutativity. In Section 5 we propose a Lie algebraic generalization of the Federbush model. In Section 6 we sustain the relation between the Federbush and the HSG-models by carrying out the ultraviolet limit. We state our conclusions in Section 7.

2. Fock space representation for the FZ-operators

In order to proceed in the way as outlined above, we have to provide explicit representations for the creation operators $Z_{\mu}^{\dagger}(\theta)$ and the fields $\mathcal{O}(x)$. The former operators are characterized by their braiding behaviour, i.e., when they are exchanged they pick up the scattering matrix as a structure constant. We restrict our considerations in this manuscript to theories in which backscattering is absent, such that the exchange algebra for the Z-operators reads [9]

$$Z_i^{\dagger}(\theta_i)Z_j^{\dagger}(\theta_j) = S_{ij}(\theta_{ij})Z_j^{\dagger}(\theta_j)Z_i^{\dagger}(\theta_i) = \exp\bigl[2\pi i\delta_{ij}(\theta_{ij})\bigr]Z_j^{\dagger}(\theta_j)Z_i^{\dagger}(\theta_i). \tag{6}$$

As indicated in (6), the scattering matrix $S_{ij}(\theta_{ij})$ can be expressed as a phase. Identical relations hold for the annihilation operators, i.e., $Z^{\dagger}(\theta) \to Z(\theta)$ in (6). When we braid a creation and an annihilation operator the presence of an additional central term was suggested in [10]

$$Z_i(\theta_i)Z_j^{\dagger}(\theta_j) = S_{ij}(\theta_{ij})Z_j^{\dagger}(\theta_j)Z_i(\theta_i) + 2\pi\delta_{ij}\delta(\theta_i - \theta_j), \tag{7}$$

which ensures that one recovers the usual (fermionic) bosonic (anti)-commutation relations in the case $(S=-1)$ $S=1$. The relations (6), (7) are commonly referred to as Faddeev–Zamolodchikov (FZ) algebra. A representation for these operators in the bosonic Fock space was first provided in [11]

$$Z_i^\dagger(\theta) = \exp\left[-i\int_\theta^\infty d\theta'\,\delta_{il}(\theta-\theta')a_l^\dagger(\theta')a_l(\theta')\right]a_i^\dagger(\theta). \tag{8}$$

By replacing a constant phase with the rapidity dependent phase $\delta_{ij}(\theta)$ and turning the expression into a convolution with an additional sum over l, the expression (8) constitutes a generalization of formulae found in the late seventies [12], which interpolate between bosonic and fermionic Fock spaces for arbitrary spin. The latter construction may be viewed as a continuous version of a Jordan–Wigner transformation [13], albeit on the lattice the commutation relations are not purely bosonic or fermionic, since certain operators anticommute at the same site but commute on different sites. Alternatively, one may also replace the bosonic a's in (8) by operators satisfying the usual fermionic anticommutation relations

$$\left\{a_i(\theta), a_j(\theta')\right\} = 0 \quad \text{and} \quad \left\{a_i(\theta), a_j^\dagger(\theta')\right\} = 2\pi\delta_{ij}\delta(\theta-\theta') \tag{9}$$

and note that the relations (6) are still satisfied [14]. In the following we want to work with this fermionic representation of the FZ-algebra (6). Having obtained a fairly simple realization for the Z-operators, we may now seek to represent the operator content of the theory in the same space. In general, this is not known and we have to resort to a study of explicit models at this stage.

3. Complex free fermions

To illustrate the procedure, to fix some of our notations and to set the scene for the Federbush model, we will commence with the free fermion. Let us consider N complex (Dirac) free Fermions described as usual by the Lagrangian density [1]

$$\mathcal{L}_{\mathrm{FF}} = \sum_{\alpha=1}^{N} \bar\psi_\alpha\left(i\gamma^\mu\partial_\mu - m_\alpha\right)\psi_\alpha. \tag{10}$$

The associated equations of motion, i.e., the Dirac equations $(i\gamma^\mu\partial_\mu - m_\alpha)\psi_\alpha = 0$, may then of course be solved with the help of the well-known Fourier decomposition of the

[1] We use the following conventions throughout the paper:

$$x^\mu = (x^0, x^1), \qquad p^\mu = (m\cosh\theta, m\sinh\theta),$$
$$g^{00} = -g^{11} = \varepsilon^{01} = -\varepsilon^{10} = 1,$$
$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \gamma^5 = \gamma^0\gamma^1,$$
$$\psi_\alpha = \begin{pmatrix} \psi_\alpha^{(1)} \\ \psi_\alpha^{(2)} \end{pmatrix}, \qquad \bar\psi_\alpha = \psi_\alpha^\dagger\gamma^0.$$

complex free Fermi fields

$$\psi_\alpha(x) = \int \frac{dp_\alpha^1}{\sqrt{4\pi}\, p_\alpha^0} \big(a_\alpha(p) u_\alpha(p) e^{-ip_\alpha\cdot x} + a_{\bar\alpha}^\dagger(p) v_\alpha(p) e^{ip_\alpha\cdot x}\big), \quad \alpha = 1,\ldots,N. \tag{11}$$

We abbreviated as usual $\sqrt{m_\alpha^2 + p_\alpha^2} = p_\alpha^0$ and employed the Weyl spinors

$$u_\alpha(p) = \sqrt{\frac{m_\alpha}{2}} \begin{pmatrix} e^{-\theta/2} \\ e^{\theta/2} \end{pmatrix} \quad \text{and} \quad v_\alpha(p) = i\sqrt{\frac{m_\alpha}{2}} \begin{pmatrix} e^{-\theta/2} \\ -e^{\theta/2} \end{pmatrix}. \tag{12}$$

The amplitudes of the scattering matrices are simply $S_{\alpha\alpha'} = -1$ for all combinations of α, α' and the FZ-algebra coincides by construction with the Clifford algebra (9), that is $Z_\alpha(\theta) = a_\alpha(\theta)$.

A further property, which we want to exploit here and in the next section, is the $U(1)$-symmetry of the Lagrangian $\mathcal{L}_{\mathrm{FF}}$, that is changing

$$\psi_\alpha(x) \to \eta_\alpha \psi_\alpha(x), \tag{13}$$

with $\eta_\alpha \in U(1)$ leaves the Lagrangian in (10) invariant. This simple symmetry will allow an a priori judgement about vanishing form factors.

3.1. Form factors of some local operators

Let us now define a prototype auxiliary field

$$\begin{aligned}\chi_\kappa^\alpha(x) = \frac{1}{4\pi^2} \int d\theta\, d\theta' \Big[&\kappa^\alpha(\theta,\theta')\big(a_\alpha^\dagger(\theta) a_{\bar\alpha}^\dagger(\theta') e^{i(p+p')\cdot x} + a_\alpha(\theta) a_{\bar\alpha}(\theta') e^{-i(p+p')\cdot x}\big) \\ &+ \kappa^\alpha(\theta, \theta' - i\pi)\big(a_{\bar\alpha}^\dagger(\theta) a_{\bar\alpha}(\theta') e^{i(p-p')\cdot x} - a_\alpha(\theta) a_\alpha^\dagger(\theta') e^{-i(p-p')\cdot x}\big)\Big].\end{aligned} \tag{14}$$

This field is essentially bilinear in the free Fermi fields up to the function $\kappa(\theta,\theta')$, whose precise expression, which gives the field its individual characteristic, we will leave generic for the time being. The properties of this function, like $\kappa(\theta - i\pi, \theta' - i\pi) = \kappa(\theta,\theta')$, as well as the form of the space–time dependent exponentials, are dictated by the crossing in (1). It means bringing consistently some of the particle creation operators $Z_\mu^\dagger(\theta)$ to the left of the operator $\mathcal{O}(0)$ introduces these constraints. Fields of this nature appear already in [16]. We now want to compute the matrix element of a general operator composed out of $\chi_\kappa^\alpha(x)$

$$\mathcal{O}^{\chi_\kappa^\alpha}(x) = {:}e^{\chi_\kappa^\alpha(x)}{:}\,. \tag{15}$$

The direct computation of matrix elements related to these fields is straightforward by employing Wick's first theorem. [2] Noting that the contribution from the normal ordered part is of course zero, since all annihilation and creation operators are brought to the left

[2] The difference between the product of some linear operators and its normal ordered product has to be a c-number determined by all possible contractions, i.e., for the linear operators $A, B, C, \ldots$ holds $ABC\cdots - {:}ABC\cdots{:} =$ sum over all possible contractions (see, e.g., [19]).

and right, respectively, we obtain for instance

$$\begin{aligned}\widetilde{F}_2^{\chi_\kappa^\alpha|\bar{\alpha}\alpha}(\theta_1,\theta_2) &= \frac{1}{4\pi^2}\int d\theta\, d\theta'\, \kappa^\alpha(\theta,\theta')\Big(a_\alpha(\theta)a_{\bar{\alpha}}(\theta')a_{\bar{\alpha}}^\dagger(\theta_1)a_\alpha^\dagger(\theta_2)\Big) \\ &= \int d\theta\, d\theta'\, \kappa^\alpha(\theta,\theta')\delta(\theta-\theta_2)\delta(\theta'-\theta_1) = \kappa^\alpha(\theta_2,\theta_1).\end{aligned} \tag{16}$$

Proceeding in this way to higher particle numbers, we compute

$$\widetilde{F}_{2n}^{\mathcal{O}^{\chi_\kappa^\alpha}|n\times\bar{\alpha}\alpha}(\theta_1,\ldots,\theta_{2n}) = \frac{1}{n!}\int d\theta'_1\cdots d\theta'_{2n}\prod_{i=1}^{n}\kappa^\alpha\big(\theta'_{2i-1},\theta'_{2i}\big)\det\mathcal{D}^{2n}, \tag{17}$$

where $\mathcal{D}^\ell$ is a rank ℓ matrix whose entries are given by

$$\mathcal{D}_{ij}^\ell = \cos^2\big[(i-j)\pi/2\big]\delta(\theta'_i-\theta_j), \quad 1\leqslant i,j\leqslant \ell. \tag{18}$$

We used the identity

$$\begin{aligned}&a_\alpha(\theta'_n)\cdots a_{\bar{\alpha}}(\theta'_3)a_\alpha(\theta'_2)a_{\bar{\alpha}}(\theta'_1)a_{\bar{\alpha}}^\dagger(\theta_1)a_\alpha^\dagger(\theta_2)a_{\bar{\alpha}}^\dagger(\theta_3)\cdots a_\alpha^\dagger(\theta_n) \\ &\quad + a_\alpha(\theta'_n)\cdots a_{\bar{\alpha}}(\theta'_3)a_\alpha(\theta'_2)a_{\bar{\alpha}}(\theta'_1)a_{\bar{\alpha}}^\dagger(\theta_1)a_\alpha^\dagger(\theta_2)a_{\bar{\alpha}}^\dagger(\theta_3)\cdots a_\alpha^\dagger(\theta_n)+\cdots = \operatorname{Det}\mathcal{D}^n.\end{aligned} \tag{19}$$

A further generic field, which we want to study and which, in contrast to $\mathcal{O}^{\chi_\kappa^\alpha}$, now possesses nonvanishing matrix elements with an odd particle number is

$$\widehat{\mathcal{O}}^{\chi_\kappa^\alpha}(x) = :\hat{\psi}_\alpha(x)e^{\chi_\kappa^\alpha(x)}:. \tag{20}$$

This field involves the fermionic field with the spinor structure stripped off

$$\hat{\psi}_\alpha(x) = \int\frac{dp_\alpha^1}{2\pi p_\alpha^0}\big(a_\alpha(p)e^{-ip_\alpha\cdot x} + a_{\bar{\alpha}}^\dagger(p)e^{ip_\alpha\cdot x}\big). \tag{21}$$

Similarly as before we compute the matrix elements

$$\widetilde{F}_{2n}^{\widehat{\mathcal{O}}^{\chi_\kappa^\alpha}|\alpha(n\times\bar{\alpha}\alpha)}(\theta_1,\ldots,\theta_{2n+1}) = \frac{1}{n!}\int d\theta'_1\cdots d\theta'_{2n+1}\prod_{i=1}^{n}\kappa^\alpha\big(\theta'_{2i},\theta'_{2i+1}\big)\det\mathcal{D}^{2n+1}. \tag{22}$$

Note that $\mathcal{O}^{\chi_\kappa^\alpha}(x)$ and $\widehat{\mathcal{O}}^{\chi_\kappa^\alpha}(x)$ are in general nonlocal operators, in the sense that it is not guaranteed that they (anti)-commute for space-like separations, i.e., $[\mathcal{O}(x),\mathcal{O}'(y)]=0$ for $(x-y)^2<0$. At the same time $\widetilde{F}_n^{\mathcal{O}}$ is just the matrix element as defined on the r.h.s. of (1) and not yet a form factor of a local field, in the sense that it satisfies the consistency equations (2)–(4), which imply locality of $\mathcal{O}$.[3] In order to distinguish between this two different situations we denote matrix elements in general by $\widetilde{F}_n^{\mathcal{O}}$ and form factors of local operators by $F_n^{\mathcal{O}}$. For instance, as a consequence of the monodromy equation (3),

[3] A rigorous proof of this statement to hold in generality is still an open issue.

a necessary condition for these two functions to coincide for $\chi_\kappa^\alpha(x)$ is

$$\kappa^\alpha(\theta, \theta' + 2\pi i) = -\gamma_{\bar{\alpha}}^{\chi_\kappa^\alpha} \kappa^\alpha(\theta, \theta'). \tag{23}$$

Before specifying the functions κ more concretely such that the corresponding $\mathcal{O}$'s become local, we would like to compare briefly the generic operators of the type (14), (15) and (20) with some general expressions for "local" operators which appear in the literature [14,17, 18]. We carry out this argument in generality without restriction to a concrete model. Let us restore in Eq. (1) the space–time dependence, multiply the equation from the left with the bra-vector $\langle Z_{\mu_n}^\dagger(\theta_n) \cdots Z_{\mu_1}^\dagger(\theta_1)|$ and introduce the necessary amount of sums and integrals over the complete states such that one can identify the identity operator $\mathbb{I}$

$$\begin{aligned}
&\sum_{\substack{n=1\cdots\infty \\ \mu_1\cdots\mu_n}} \int_{-\infty}^{\infty} \frac{d\theta_1 \cdots d\theta_n}{n!(2\pi)^n} F_n^{\mathcal{O}|\mu_1\cdots\mu_n}(\theta_1 \cdots \theta_n) \langle Z_{\mu_n}^\dagger(\theta_n) \cdots Z_{\mu_1}^\dagger(\theta_1)| e^{-i\sum_j p_j \cdot x} \\
&\quad = \sum_{\substack{n=1\cdots\infty \\ \mu_1\cdots\mu_n}} \int_{-\infty}^{\infty} \frac{d\theta_1 \cdots d\theta_n}{n!(2\pi)^n} \langle \mathcal{O}(x) Z_{\mu_1}^\dagger(\theta_1) \cdots Z_{\mu_n}^\dagger(\theta_n)\rangle \langle Z_{\mu_n}^\dagger(\theta_n) \cdots Z_{\mu_1}^\dagger(\theta_1)| \\
&\quad = \langle \mathcal{O}(x)| \mathbb{I}.
\end{aligned}$$

Cancelling the vacuum in the first and last line, and noting that we can replace the product of operators, which is left over also by its normal ordered version, we obtain the expression defined originally in [17]

$$\widetilde{\mathcal{O}}(x) = \sum_{\substack{n=1\cdots\infty \\ \mu_1\cdots\mu_n}} \int_{-\infty}^{\infty} \frac{d\theta_1 \cdots d\theta_n}{n!(2\pi)^n} F_n^{\mathcal{O}|\mu_1\cdots\mu_n}(\theta_1 \cdots \theta_n) :Z_{\mu_n}^\dagger(\theta_n) \cdots Z_{\mu_1}^\dagger(\theta_1): e^{-i\sum_j p_j \cdot x} \tag{24}$$

Hence this field is simply an inversion of (1). From its very construction it is clear that $\widetilde{\mathcal{O}}(x)$ is a meaningful field in the weak sense, that is acting on an in-state we will recover by construction the form factor related to $\mathcal{O}(x)$. In addition, one may also construct the well-known expression of the two-point correlation function expanded in terms of form factors, as stated in [17]. However, it is also clear that $\widetilde{\mathcal{O}}(x) \neq \mathcal{O}(x)$, simply by comparing (24) and the explicit expressions for some local fields occurring in the free fermionic theory, e.g., (14), (15) and (20). The reason is that acting on an in-state with the latter expressions the form factors are generated in a nontrivial Wick contraction procedure, whereas when doing the same with (24) the Wick contractions will be trivial. Therefore, general statements and conclusions drawn from an analysis made on $\widetilde{\mathcal{O}}(x)$ should be taken with care. It is also needless to say that from a practical point of view the expression (24) is rather empty, since the expressions of the form factors $F_n^{\mathcal{O}|\mu_1\cdots\mu_n}(\theta_1 \cdots \theta_n)$ themselves are usually not known and their determination is in general a quite nontrivial task. In [14,17,18] the integration in the formula (24) is a rather artificial contour integration which takes care about analytic continuations of values of $i\pi$. This does not seem to be a fundamental feature, since it remains completely obscure how to incorporate bound states in this manner.

Let us now return to our concrete analysis by specifying κ.

3.1.1. The order and disorder field

Having in mind to proceed to the Federbush model, we will restrict ourselves from now on to the case of two complex fermions, i.e., $N=2$ in the Lagrangian (10). The free fermionic theory possesses some very distinct fields, namely the disorder and order fields

$$\mu_\alpha(x) = {:}e^{\omega_\alpha(x)}{:} \quad \text{and} \quad \sigma_\alpha(x) = {:}\hat{\psi}_\alpha(x)\mu_\alpha(x){:}\,, \quad \alpha = 1, 2, \tag{25}$$

respectively. The names for these fields result from the ultraviolet limit, see also Section 5, since then they flow to their equivalent counterparts in the conformal field theory [15], namely, to primary fields with scaling dimension $1/16$. We introduced here the fields

$$\omega_\alpha(x) = \chi_\kappa^\alpha(x) \quad \text{with} \quad \kappa^1(\theta,\theta') = -\kappa^2(-\theta,-\theta') = \frac{i}{2}\frac{e^{-\frac{1}{2}(\theta-\theta')}}{\cosh\frac{1}{2}(\theta-\theta')}. \tag{26}$$

Admittedly, the precise form of the fields $\omega_\alpha(x)$ appears to be slightly unmotivated at this stage. However, we will provide a better rational for this in the next section, where we see that they originate by relating a so-called triple normal ordering procedure for a field, which can be constructed directly from the Fourier decomposition of the free Fermi fields (11), to another one associated with the usual Wick normal ordering. It will turn out that the field $\omega_\alpha(x)$ emerges as the limit of a Federbush model field to the free fermionic theory, i.e., $\lim_{\lambda\to 1/2}\Omega_\alpha^\lambda(x) = \omega_\alpha(x)$, see Eq. (71).

Let us now compute the form factors related to the above mentioned fields $\mu_\alpha(x)$ and $\sigma_\alpha(x)$. Using the particular form of $\kappa^\alpha(\theta,\theta')$ as defined in (26), we compute the integrals in (17) and obtain a closed expression for the n-particle form factors of the disorder operators

$$\begin{aligned}
F_{2n}^{\mu_1|n\times\bar{1}1}(\theta_1,\ldots,\theta_{2n}) &= (-1)^n F_{2n}^{\mu_2|n\times\bar{2}2}(-\theta_1,\ldots,-\theta_{2n}),\\
F_{2n}^{\mu_{\bar{1}}|n\times\bar{1}1}(-\theta_1,\ldots,-\theta_{2n}) &= (-1)^n F_{2n}^{\mu_{\bar{2}}|n\times\bar{2}2}(\theta_1,\ldots,\theta_{2n})\\
&= i^n 2^{n-1}\sigma_n(\bar{x}_1,\bar{x}_3,\ldots,\bar{x}_{2n-1})\mathcal{B}_{n,n},
\end{aligned} \tag{27}$$

with

$$\begin{aligned}
\mathcal{B}_{n,m} &= \frac{\prod_{1\leqslant i<j\leqslant n}(\bar{x}_{2i-1}^2 - \bar{x}_{2j-1}^2)\prod_{1\leqslant i<j\leqslant m}(x_{2i}^2 - x_{2j}^2)}{\prod_{1\leqslant i<j\leqslant n+m}(u_i+u_j)}\\
&= \frac{\det\mathcal{V}^m(x^2)\det\mathcal{V}^n(\bar{x}^2)}{\det\mathcal{W}^{n+m}(u)}.
\end{aligned} \tag{28}$$

Associated with the particles and antiparticles we introduced here the quantities $x_i = \exp(\theta_i)$ and $\bar{x}_i = \exp(\theta_i)$, respectively. The variable u_i can be either of them. We also employed the elementary symmetric polynomials $\sigma_k(x_1,\ldots,x_n)$, defined as

$$\sigma_k(x_1,\ldots,x_n) = \sum_{l_1<\cdots<l_k} x_{l_1}\cdots x_{l_k} \tag{29}$$

(see, e.g., [21] for more properties), the Vandermonde determinant of the rank ℓ matrix $\mathcal{V}^\ell$ whose entries are given by

$$\mathcal{V}_{ij}^\ell(x) = (x_j)^{i-1}, \quad 1\leqslant i,j\leqslant\ell \tag{30}$$

and the determinant of the rank $\ell-1$ matrix $\mathcal{W}^{\ell-1}$ with entries

$$\mathcal{W}_{ij}^{\ell-1}(x)=\sigma_{2i-j}(x_1,\ldots,x_\ell),\quad 1\leqslant i,j\leqslant \ell-1. \tag{31}$$

The relations between $F_{2n}^{\mu_1|n\times\bar{1}1}$ and $F_{2n}^{\mu_2|n\times\bar{2}2}$ as stated in (27) follow most transparently from (26) and (17). One may easily verify that the expression (27) indeed satisfies the consistency equations (2)–(4) with $\gamma_{\bar{\alpha}}^{\mu_\alpha}=-1$ for $\alpha=1,2$. We justify this choice in the next section by carrying out the Federbush model $\rightarrow$ two complex free fermion limit. Noting that $\mu_\alpha(x)$ is invariant with respect to the symmetry property (13), it follows immediately that

$$F_{k+l}^{\mu_{\alpha'}|\alpha\alpha\cdots\alpha\alpha\beta\beta\cdots\beta\beta}(\theta_1,\ldots,\theta_k,\theta_{k+1},\ldots,\theta_{k+l})=0, \tag{32}$$

for $\alpha\neq\bar{\beta},\ k\neq l,\ \alpha'=1,2,\bar{1},\bar{2}$. This means that, up to a reordering of the particles, the expressions reported in (27) are in fact the only nonvanishing form factors related to $\mu_\alpha(x)$ for $\alpha\in\{1,2,\bar{1},\bar{2}\}$.

In a similar way we compute the n-particle form factors of the order operator

$$\begin{aligned}
&F_{2n+1}^{\sigma_1|1(n\times\bar{1}1)}(\theta_1,\ldots,\theta_{2n+1})=(-1)^n F_{2n+1}^{\sigma_2|2(n\times\bar{2}2)}(-\theta_1,\ldots,-\theta_{2n+1}),\\
&F_{2n+1}^{\sigma_{\bar{1}}|1(n\times\bar{1}1)}(-\theta_1,\ldots,-\theta_{2n+1})=(-1)^n F_{2n+1}^{\sigma_{\bar{2}}|2(n\times\bar{2}2)}(\theta_1,\ldots,\theta_{2n+1})\\
&\qquad\qquad=i^n 2^{n-1}\sigma_n(\bar{x}_1,\ldots,\bar{x}_{2n-1})\mathcal{B}_{n,n+1}.
\end{aligned} \tag{33}$$

As a consistency check, one may once again verify that (33) fulfills the form factor equations (2)–(4) with $\gamma_{\bar{\alpha}}^{\sigma_\alpha}=1$ for $\alpha=1,2$. Again, we postpone the justification of this choice to the next section by carrying out the Federbush model $\rightarrow$ two complex free fermion limit. Noting that $\sigma_\alpha(x)\rightarrow\eta_\alpha\sigma_\alpha(x)$ by (13), it follows immediately that

$$F_{k+l}^{\sigma_\alpha|\alpha\alpha\cdots\alpha\alpha\beta\beta\cdots\beta\beta}(\theta_1,\ldots,\theta_k,\theta_{k+1}\ldots,\theta_{k+l})=0\quad \text{for } \alpha\neq\bar{\beta},\ k\neq l+1. \tag{34}$$

Of course, this way of proceeding also works for the real free Fermion and one may recover the well-known expressions of the literature [3,16,20]. As a difference to our previous computations, however, we have to take care of more contributions in the contraction procedure. Keeping the form of $\kappa^\alpha(\theta,\theta')$ as defined in (26), but taking $\alpha=\bar{\alpha}$, we compute for instance

$$\begin{aligned}
F_2^{\mu}(\theta_1,\theta_2)&=\frac{1}{4\pi^2}\int d\theta\, d\theta'\,\kappa(\theta,\theta')\\
&\qquad\times\Big(a(\theta)a(\theta')a^\dagger(\theta_1)a^\dagger(\theta_2)+a(\theta)a(\theta')a^\dagger(\theta_1)a^\dagger(\theta_2)\Big)\\
&=\int d\theta\, d\theta'\,\kappa(\theta,\theta')\big[\delta(\theta-\theta_2)\delta(\theta'-\theta_1)-\delta(\theta-\theta_1)\delta(\theta'-\theta_2)\big]\\
&=i\tanh\frac{\theta_{12}}{2}.
\end{aligned} \tag{35}$$

Proceeding in this way to the higher n-particle form factors we only have to replace in (17) the matrix $\mathcal{D}^\ell$ with $\widetilde{\mathcal{D}}^\ell$, whose entries are $\widetilde{\mathcal{D}}_{ij}^\ell=\delta(\theta_i'-\theta_j)$. Computing the integrals we get

$$F_{2n}^{\mu}(\theta_1,\ldots,\theta_{2n}) = i^n\,\mathrm{Pf}(\mathcal{A}) = i^n\sqrt{\det\mathcal{A}} = i^n \prod_{1\leqslant i,j\leqslant 2n} \tanh\frac{\theta_{ij}}{2}, \tag{36}$$

where $\mathcal{A}$ is an antisymmetric $(2n\times 2n)$-matrix whose entries are given by $\mathcal{A}_{ij} = \tanh\theta_{ij}/2$ and Pf denotes its Pfaffian. [4] In a similar way we compute the n-particle form factors of the order operator

$$F_{2n+1}^{\sigma}(\theta_1,\ldots,\theta_{2n+1}) = i^n\,\mathrm{Pf}(\mathcal{A}) = i^n \prod_{1\leqslant i,j\leqslant 2n+1} \tanh\frac{\theta_{ij}}{2}. \tag{37}$$

Expressions of the type (36) and (37) can be found already in the first paper of [16]. The product expressions for F_{2n}^{μ} and F_{2n+1}^{σ} were also derived in [20] and [3], respectively, by means of solving the form factor consistency equations (2)–(4).

3.1.2. The energy–momentum tensor

A further field which plays an important role in any theory is the energy–momentum tensor, which for the free fermion in our normalization simply reads

$$T^{\mu}{}_{\nu} = 2i\left(:\bar{\psi}_1\gamma^{\mu}\partial_{\nu}\psi_1: + :\bar{\psi}_2\gamma^{\mu}\partial_{\nu}\psi_2:\right). \tag{38}$$

With the help of Eq. (11) we compute easily

$$T^{\mu\nu} = \chi_{t_1^{\mu\nu}} + \chi_{t_2^{\mu\nu}}, \tag{39}$$

using

$$\begin{aligned} t_{\alpha}^{0\mu}(\theta,\tilde{\theta}) &= -2\pi i m_{\alpha}(p_{\alpha})^{\mu}\sinh\frac{\theta+\tilde{\theta}}{2}, \\ t_{\alpha}^{1\mu}(\theta,\tilde{\theta}) &= -2\pi i m_{\alpha}(p_{\alpha})^{\mu}\cosh\frac{\theta+\tilde{\theta}}{2}, \end{aligned} \tag{40}$$

where we recall the definition of $\chi_{\kappa}(x)$ from (14). A specially distinct role is played by the trace of the energy–momentum tensor, since on one hand it is directly proportional to the operator which breaks the conformal invariance [22] and on the other hand it occurs explicitly in various computations associated to the ultraviolet limit like the c-theorem [23] and the Δ-sum rule [24] (see Section 5). It acquires the explicit form

$$\begin{aligned} T^{\mu}{}_{\mu} &= 2im_1:\bar{\psi}_1\psi_1: + 2im_2:\bar{\psi}_2\psi_2: = \chi_{t_1} + \chi_{t_2}, \\ t_{\alpha}(\theta,\tilde{\theta}) &= 2\pi i m_{\alpha}^2\sinh\frac{\tilde{\theta}-\theta}{2}. \end{aligned} \tag{41}$$

It is clear that only the two-particle form factor can be different from zero and we compute it in an analogous way as in the previous section, that is using Fourier decomposition (11)

[4] Denoting the permutation group of $2n$ indices by S_{2n} and the signature of the permutation π by $\mathrm{sgn}(\pi)$, the Pfaffian of a matrix $\mathcal{A}$ is defined as

$$\mathrm{Pf}(\mathcal{A}) = \frac{1}{2^n n!}\sum_{\pi\in S_{2n}} \mathrm{sgn}(\pi)\prod_{i=1}^{n} \mathcal{A}_{\pi(2i-1),\pi(2i)}.$$

with subsequent contractions,

$$F_2^{T^{0\mu}|\bar{\alpha}\alpha}(\theta,\tilde{\theta}) = -2\pi i m_\alpha p^\mu \sinh\frac{\theta+\tilde{\theta}}{2}, \tag{42}$$

$$F_2^{T^{1\mu}|\bar{\alpha}\alpha}(\theta,\tilde{\theta}) = -2\pi i m_\alpha p^\mu \cosh\frac{\theta+\tilde{\theta}}{2}, \tag{43}$$

$$F_2^{T^{\mu}{}_{\mu}|\bar{\alpha}\alpha}(\theta,\tilde{\theta}) = F_2^{T^{\mu}{}_{\mu}|\alpha\bar{\alpha}}(\theta,\tilde{\theta}) = -2\pi i m_\alpha^2 \sinh\frac{\theta-\tilde{\theta}}{2}. \tag{44}$$

When taking $\alpha = \bar{\alpha}$, these expressions coincide with the ones which may be found in the literature for the real fermion. As usual, we may verify that various equations which hold for the operators themselves also hold for the associated form factors. For instance, the conservation of the energy–momentum tensor

$$\partial_\mu T^{\mu\nu} = i\big[\widehat{P}_\mu, T^{\mu\nu}\big] = 0 \tag{45}$$

is reflected by the fact that

$$\big(p^0+\tilde{p}^0\big)F_2^{T^{\mu 0}|\bar{\alpha}\alpha} = -\big(p^1+\tilde{p}^1\big)F_2^{T^{\mu 1}|\bar{\alpha}\alpha}. \tag{46}$$

Here we used the explicit form of the momentum operator

$$\widehat{P}_\mu = \int_{-\infty}^{\infty} dx^1\, T^0{}_\mu = \sum_{\alpha=1}^{2}\int \frac{dp_\alpha^1}{2\pi p_\alpha^0}(p_\alpha)_\mu\big(a_\alpha^\dagger(p)a_\alpha(p) - a_{\bar{\alpha}}(p)a_{\bar{\alpha}}^\dagger(p)\big) \tag{47}$$

when changing in (45) derivatives to commutators by means of the Heisenberg equation of motion. It is then easy to verify that $[\widehat{P}_\mu, a_\alpha^\dagger(p)] = (p_\alpha)_\mu a_\alpha^\dagger(p)$ and $[\widehat{P}_\mu, a_\alpha(p)] = -(p_\alpha)_\mu a_\alpha(p)$, such that we verify explicitly

$$\partial_\mu \chi_\kappa^\alpha(x) = i\big[\widehat{P}_\mu, \chi_\kappa^\alpha(x)\big], \tag{48}$$

which is of course what we expect. Eq. (48) is a further support for the consistency of the generic definition of $\chi_\kappa^\alpha(x)$ in (14).

4. The Federbush model

The Federbush model [25] was proposed forty years ago as a prototype for an exactly solvable quantum field theory which obeys the Wightman axioms [26–28]. Formally it is closely related to the massive Thirring model [29]. It contains two different massive particles Ψ_1 and Ψ_2. A special feature of this model is that the related vector currents $J_\alpha^\mu = \overline{\Psi}_\alpha\gamma^\mu\Psi_\alpha$, $\alpha\in\{1,2\}$, whose analogues occur squared in the massive Thirring model, enter the Lagrangian density of the Federbush model in a parity breaking manner

$$\mathcal{L}_{\mathrm{F}} = \sum_{\alpha=1,2}\overline{\Psi}_\alpha\big(i\gamma^\mu\partial_\mu - m_\alpha\big)\Psi_\alpha - 2\pi\lambda\varepsilon_{\mu\nu}J_1^\mu J_2^\nu \tag{49}$$

due to the presence of the Levi-Civita pseudotensor ε. It is then easy to verify that the related equations of motion

$$\begin{aligned}(i\gamma^{\mu}\partial_{\mu}-m_1)\Psi_1 &= 2\pi\lambda\varepsilon_{\mu\nu}J_2^{\nu}\gamma^{\mu}\Psi_1,\\ (i\gamma^{\mu}\partial_{\mu}-m_2)\Psi_2 &= 2\pi\lambda\varepsilon_{\nu\mu}J_1^{\nu}\gamma^{\mu}\Psi_2,\end{aligned} \tag{50}$$

can be solved by

$$\begin{aligned}\Psi_1 &= \vdots\exp\big(2\sqrt{\pi}\,i\lambda\phi_2\big)\vdots\,\psi_1=\Phi_2^{\lambda}\psi_1,\\ \Psi_2 &= \vdots\exp\big(-2\sqrt{\pi}\,i\lambda\phi_1\big)\vdots\,\psi_2=\Phi_1^{\lambda}\psi_2,\end{aligned} \tag{51}$$

if in addition the free bosonic fields ϕ_α constitute potentials for axial vector currents composed out of the free fermions ψ_α

$$\frac{1}{\sqrt{\pi}}\partial_{\mu}\phi_{\alpha}=\varepsilon_{\nu\mu}J_{\alpha}^{\nu}=\bar{\psi}_{\alpha}\gamma_{\mu}\gamma^{5}\psi_{\alpha},\quad \lambda\neq 0,\ \alpha=1,2. \tag{52}$$

The triple normal ordering in Eq. (51) is defined as $\vdots e^{\kappa\phi}\vdots = e^{\kappa\phi}/\langle e^{\kappa\phi}\rangle$ for κ being some constant. This is very advantageous in the calculation of commutation relations, since one can simply deal with ordinary operator relations instead of having to handle messy Wick contractions. We stress that in case the coupling constant λ vanishes, that is when $\mathcal{L}_{\mathrm{F}}$ reduces to $\mathcal{L}_{\mathrm{FF}}$ and the relations (50) correspond to two decoupled Dirac equations, the relation (52) does not hold.

In order to compute the factors of local commutativity $\gamma_{\mu}^{\mathcal{O}}$, as defined in (5), we need various (anti)-commutation relations. The fields $\psi_\alpha(x)$ are complex free (Dirac) fermions of masses m_α and the fields $\phi_\alpha(x)$ are free bosons, such that for $\alpha,\beta=1,2$ we trivially have

$$\big[\phi_{\alpha}(x),\phi_{\beta}(y)\big]=\big[\Phi_{\alpha}(x),\Phi_{\beta}(y)\big]=\big[\phi_{\alpha}(x),\Phi_{\beta}(y)\big]=\big\{\psi_{\alpha}(x),\psi_{\beta}(y)\big\}=0, \tag{53}$$

$$\big\{\psi_{\alpha}(x),\psi_{\beta}^{\dagger}(y)\big\}=\delta_{\alpha\beta}\delta\big(x^1-y^1\big). \tag{54}$$

The commutation relations involving mixed expressions of ψ_α and ϕ_β are less obvious and in fact it is crucial to note that these fields are not mutually local, that is $[\psi_\alpha(x),\phi_\beta(y)]\neq 0$ for space-like separations, i.e., $(x-y)^2<0$. Concretely we have the following equal time exchange relations for $\alpha,\beta=1,2$

$$\big[\psi_{\alpha}(x),\phi_{\beta}(y)\big]=\sqrt{\pi}\,\delta_{\alpha\beta}\Theta\big(x^1-y^1\big)\psi_{\alpha}(x), \tag{55}$$

$$\psi_{\alpha}(x)\Phi_{\beta}^{\lambda}(y)=\Phi_{\beta}^{\lambda}(y)\psi_{\alpha}(x)e^{2\pi i(-1)^{\beta}\lambda\delta_{\alpha\beta}\Theta(x^1-y^1)}, \tag{56}$$

$$-\psi_{\alpha}(x)\Psi_{\beta}(y)=\Psi_{\beta}(y)\psi_{\alpha}(x)e^{-2\pi i(-1)^{\beta}\lambda\delta_{|\alpha-\beta|,1}\Theta(x^1-y^1)}, \tag{57}$$

$$\Psi_{\alpha}(x)\Phi_{\beta}^{\lambda}(y)=\Phi_{\beta}^{\lambda}(y)\Psi_{\alpha}(x)e^{2\pi i(-1)^{\beta}\lambda\delta_{\alpha\beta}\Theta(x^1-y^1)}, \tag{58}$$

$$-\Psi_{\alpha}(x)\Psi_{\beta}(y)=\Psi_{\beta}(y)\Psi_{\alpha}(x)e^{-2\pi i\lambda(-1)^{\beta}\delta_{|\alpha-\beta|,1}}. \tag{59}$$

We used here the Heavyside step function $\Theta(x)$, defined as usual as $\Theta(x>0)=1$, $\Theta(x<0)=0$ and $\Theta(0)=1/2$. One may convince oneself easily that (55) is compatible with (52) and that the remaining equations are straightforward consequences of (53)–(55). Apart from this choice, which agrees with the one in [32], one can also find in some places of the literature, e.g., [31], that in (55) the Θ-functions are replaced by $\varepsilon(x)/2 = \Theta(x) - \Theta(-x)$. This is of course also compatible with (52). However, an immediate consequence of our choice is that the Federbush fields $\Psi_\alpha(x)$ are only mutually local if they are of the same type α, whereas when taking the ε-function instead, they are mutually local for all values of α and β. The different choices will of course lead to different factors of local commutativity γ and will, therefore, alter the consistency equations (2)–(4). Arguing on the properties of these equations we provide more reasoning for our choice below.

A further important implication of the fact that ψ_α and ϕ_β are not mutually local is that the fields Ψ_α are in different Borchers classes [5] as the free fermion. Thus, there is a chance for the existence of a nontrivial scattering matrix, which was indeed found in [26,31]. In fact, we will now demonstrate that this S-matrix can be obtained as a limit of a more complex model, that is the homogeneous sine-Gordon (HSG) model.

4.1. Federbush models from HSG-models

Ever since the equivalence between the massive Thirring- and the sine-Gordon model was demonstrated [33], there have been various identifications between different types of models. In a similar spirit we also want to show now how a fermionic model is obtainable from a bosonic one, albeit in contrast to the above situation our fermionic scattering matrix will be constructed solely out of the asymptotic phases of a given scattering matrix

$$\lim_{\theta \to \pm\infty} S_{ij}(\theta) = e^{\Delta_{ij}^{\pm}}. \tag{60}$$

Starting from a consistent solution to the crossing and unitarity relations

$$S_{ab}^{jk}(\theta) S_{ba}^{kj}(-\theta) = 1 \quad \text{and} \quad S_{ab}^{jk}(\theta) = S_{\bar{b}a}^{\bar{k}j}(i\pi - \theta), \tag{61}$$

one clearly has the constraint $\Delta_{ij}^{+} = -\Delta_{ji}^{-} = \Delta_{\bar{j}i}^{-}$. This means that

$$\widehat{S}_{ij} = e^{\Delta_{ij}^{+} + \Delta_{ij}^{-}} \tag{62}$$

will also be a valid solution to the unitarity-crossing relations for S. Having no rapidity dependence there is no bound state bootstrap equation to be concerned about, such that (62) already constitutes a consistent scattering matrix. Concretely we will now show that when taking $S_{ij}(\theta)$ in (60) to be the scattering matrix of the $SU(3)_3$-HSG model, as found in [34], the resulting S-matrix $\widehat{S}_{ij}$ in (62) will be the one of the Federbush model at a particular value of the coupling constant. In fact it can be shown that this prescription leads to a much wider range of scattering matrices which can be directly associated to a Lagrangian of a Federbush model generalized in a Lie algebraic manner, see Section 5. Let

[5] An equivalence class of complete, local field systems is referred to as a Borchers class. Its crucial property is that it characterizes completely the scattering matrix without having to resort to particular fields. (For more details see, e.g., [30] p. 104, however, this notion is of no further relevance for our concrete computations.)

us recall now the scattering matrix of the $SU(3)_3$-HSG model

$$S^{SU(3)_3}(\theta) = \begin{pmatrix} (2)_\theta & -(1)_\theta & -(-2)_\theta e^{-i\pi\tau} & (-1)_\theta e^{i\pi\tau} \\ -(1)_\theta & (2)_\theta & (-1)_\theta e^{i\pi\tau} & -(-2)_\theta e^{-i\pi\tau} \\ -(-2)_\theta e^{i\pi\tau} & (-1)_\theta e^{-i\pi\tau} & (2)_\theta & -(1)_\theta \\ (-1)_\theta e^{-i\pi\tau} & -(-2)_\theta e^{i\pi\tau} & -(1)_\theta & (2)_\theta \end{pmatrix}. \tag{63}$$

We abbreviated $(x)_\theta = \sinh\frac{1}{2}(\theta + i\pi x/3)/\sinh\frac{1}{2}(\theta - i\pi x/3)$ and $\tau = \pm 1/3$. For the rows and columns we adopt here the ordering $\{1, \bar{1}, 2, \bar{2}\}$. We also took the resonance parameters σ of the HSG-model to be zero, since they will not play any role in our further considerations. Computing now the limit according to the above prescription we obtain

$$\lim_{\theta\to\infty}\left[S_{ij}^{SU(3)_3}(\theta)S_{ij}^{SU(3)_3}(-\theta)\right] = S^{\mathrm{FB}} = -\begin{pmatrix} 1 & 1 & e^{-2\pi i\lambda} & e^{2\pi i\lambda} \\ 1 & 1 & e^{2\pi i\lambda} & e^{-2\pi i\lambda} \\ e^{2\pi i\lambda} & e^{-2\pi i\lambda} & 1 & 1 \\ e^{-2\pi i\lambda} & e^{2\pi i\lambda} & 1 & 1 \end{pmatrix}. \tag{64}$$

We found it convenient to relate the parameter τ to the λ in the Lagrangian density (49) as $\tau = 1 - \lambda$. Then S^{FB} corresponds to the scattering matrix derived in [26,31], apart from the overall minus sign, which is due to the fact that we adopt the convention that the particles are ordered in opposite order in the in- and out-states, i.e., we include the statistics factor into the S-matrix. After having taken the limit (64), the crossing and unitarity equations also hold when we relax the constraint for τ and allow it to take completely generic values different from 1/3. Thus, whenever the coupling constant λ becomes an even integer the theory decouples into a system of two free complex fermions. From a Lagrangian point of view we expect this kind of behaviour of course for vanishing λ.

Having specified the scattering matrix of the model, we are in the position to state directly from (8) a representation for the FZ-algebra. The explicit version of (8) then reads

$$Z_1^\dagger(\theta) = \exp\left(-i\lambda\int_\theta^\infty d\theta' :\rho_2(\theta'):\right)a_1^\dagger(\theta), \tag{65}$$

$$Z_{\bar{1}}^\dagger(\theta) = \exp\left(i\lambda\int_\theta^\infty d\theta' :\rho_2(\theta'):\right)a_{\bar{1}}^\dagger(\theta), \tag{66}$$

$$Z_2^\dagger(\theta) = \exp\left(i\lambda\int_\theta^\infty d\theta' :\rho_1(\theta'):\right)a_2^\dagger(\theta), \tag{67}$$

$$Z_{\bar{2}}^\dagger(\theta) = \exp\left(-i\lambda\int_\theta^\infty d\theta' :\rho_1(\theta'):\right)a_{\bar{2}}^\dagger(\theta), \tag{68}$$

with

$$\rho_\alpha(\theta) = a_\alpha^\dagger(\theta)a_\alpha(\theta) - a_{\bar{\alpha}}^\dagger(\theta)a_{\bar{\alpha}}(\theta). \tag{69}$$

We will now specify more concretely various local operators of the Federbush model for which we want to compute the form factors explicitly by using the fermionic free field representation (65)–(68).

4.2. *Form factors of some local operators*

We compute now explicitly the bosonic fields $\phi_\alpha(x)$ by solving equation (52) and express them in terms of our general formula (14)

$$\phi_\alpha(x) = \sqrt{\pi} \int_{-\infty}^{x^1} dx^1 :\Psi_\alpha^\dagger \Psi_\alpha: = \chi_{\tilde{\kappa}}^\alpha(x) \quad \text{with} \quad \tilde{\kappa}^\alpha(\theta, \theta') = \frac{\pi^{\frac{3}{2}}}{2\cosh\frac{1}{2}(\theta - \theta')}. \tag{70}$$

A field closely related to $\phi_\alpha(x)$, but whose origin is far less direct, is

$$\Omega_\alpha^\lambda(x) = \chi_{\hat{\kappa}}^\alpha(x) \quad \text{with} \quad \hat{\kappa}^1(\theta, \theta') = -\hat{\kappa}^2(-\theta, -\theta') = \frac{i \sin(\pi\lambda) e^{-\lambda(\theta-\theta')}}{2\cosh\frac{1}{2}(\theta - \theta')}. \tag{71}$$

It is this field which constitutes the analogue to the auxiliary field already used in the previous section. In view of the periodicity of the scattering matrix (49), we may restrict the range of λ to $\lambda \in (0, 2)/1$. The special role of $\lambda = 1$ was treated in more detail in [35]. Important for our purposes is the value $\lambda = 1/2$ for which the operator $\Omega_\alpha^\lambda(x)$ reduces to $\omega_\alpha(x)$ as defined in Eq. (14).

4.2.1. *The order and disorder field*

In close relation to the free fermionic theory one may also introduce the analogue fields to the disorder and order fields in the Federbush model

$$\Phi_\alpha^\lambda(x) = :\exp\left[\Omega_\alpha^\lambda(x)\right]: \quad \text{and} \quad \Sigma_\alpha^\lambda(x) = :\hat{\psi}_\alpha(x)\Phi_\alpha^\lambda(x):. \tag{72}$$

In [32], Lehmann and Stehr showed the remarkable fact that the operator $\Phi_\alpha^\lambda(x)$, which is composed out of free Bosons, occurring in (51) can be viewed in two equivalent ways. On one hand it can be defined through a so-called triple ordered product and on the other hand by means of a conventional fermionic Wick ordered expression

$$\Phi_\alpha^\lambda(x) = \vdots \exp\left[-2\sqrt{\pi}\, i\lambda\phi_\alpha(x)\right] \vdots = :\exp\left[\Omega_\alpha^\lambda(x)\right]:. \tag{73}$$

Having again in mind to compute the factors of local commutativity $\gamma_\mu^{\mathcal{O}}$, as defined in (5), we need various equal time exchange relations. With the help of (55)–(59) we compute

$$-\psi_\alpha(x)\Sigma_\beta^\lambda(y) = \Sigma_\beta^\lambda(y)\psi_\alpha(x) e^{2\pi i(-1)^\beta \lambda \delta_{\alpha\beta} \Theta(x^1 - y^1)}, \tag{74}$$

$$\Phi_\alpha^\lambda(x)\Sigma_\beta^\lambda(y) = \Sigma_\beta^\lambda(y)\Phi_\alpha^\lambda(x) e^{2\pi i(-1)^\beta \lambda \delta_{\alpha\beta} \Theta(x^1 - y^1)}, \tag{75}$$

$$-\Psi_\alpha(x)\Sigma_\beta^\lambda(y) = \Sigma_\beta^\lambda(y)\Psi_\alpha(x) e^{2\pi i\lambda(-1)^\beta (\delta_{\alpha\beta}\Theta(x^1 - y^1) - \delta_{|\alpha-\beta|,1}\Theta(y^1 - x^1))}, \tag{76}$$

$$\Sigma_\alpha^\lambda(x)\Sigma_\beta^\lambda(y) = \Sigma_\beta^\lambda(y)\Sigma_\alpha^\lambda(x) e^{2\pi i(-1)^\beta \lambda \delta_{\alpha\beta}}. \tag{77}$$

Having obtained the relevant exchange relations we can read off the factors of local commutativity for the operators under consideration

$$\gamma_\alpha^{\Phi_\beta^\lambda} = -\gamma_\alpha^{\Sigma_\beta^\lambda} = e^{2\pi i(-1)^\beta \lambda \delta_{\alpha\beta}} \quad \text{and} \quad \gamma_{\bar{\alpha}}^{\Phi_\beta^\lambda} = -\gamma_{\bar{\alpha}}^{\Sigma_\beta^\lambda} = e^{-2\pi i(-1)^\beta \lambda \delta_{\alpha\beta}}. \tag{78}$$

Note in particular, that for $\lambda \to 1/2$ we recover, as we expect, the values corresponding to the two complex free fermions

$$\lim_{\lambda\to 1/2} \gamma_\alpha^{\Phi_\alpha^\lambda} = \gamma_\alpha^{\mu_\alpha} = -1 \quad \text{and} \quad \lim_{\lambda\to 1/2} \gamma_\alpha^{\Sigma_\alpha^\lambda} = \gamma_\alpha^{\sigma_\alpha} = 1. \tag{79}$$

Having assembled all the ingredients, let us now turn to the explicit computation of the n-particle form factors related to the field $\Phi_\alpha^\lambda(x)$. Since $\mathcal{L}_{\mathrm{FF}}$ respects the same symmetry as $\mathcal{L}_{\mathrm{F}}$, namely, (13), it is an immediate consequence that the only nonvanishing form factors of $\Phi_\alpha^\lambda(x)$ have to involve an equal number of particles and antiparticles α and $\bar{\alpha}$. That means

$$F_{k+l}^{\Phi_{\alpha'}^\lambda|\alpha\alpha\cdots\alpha\alpha\beta\beta\cdots\beta\beta}(\theta_1,\ldots,\theta_k,\theta_{k+1},\ldots,\theta_{k+l}) = 0 \tag{80}$$

$\alpha \neq \bar{\beta}$, $k \neq l$, $\alpha' = 1, 2, \bar{1}, \bar{2}$. Turning now to the nonvanishing form factors, we compute by employing again Wick's theorem

$$\begin{aligned} F_2^{\Phi_1^\lambda|\bar{1}1}(\theta_1,\theta_2) &= \frac{1}{4\pi^2}\int d\theta\, d\theta'\, \kappa^\alpha(\theta,\theta')\big(a_\alpha(\theta) a_{\bar{\alpha}}(\theta') Z_{\bar{\alpha}}^\dagger(\theta_1) Z_\alpha^\dagger(\theta_2)\big) \\ &= \frac{i\sin(\pi\lambda)e^{\lambda\theta_{12}}}{2\cosh\frac{1}{2}\theta_{12}} = F_2^{\Phi_2^{-\lambda}|\bar{2}2}(\theta_1,\theta_2). \end{aligned} \tag{81}$$

Note, that in the contraction of an $a_\alpha(\theta)$- and a $Z_\alpha^\dagger(\theta)$-operator there is no contribution from the exponential term inside the $Z_\alpha^\dagger(\theta)$, since it involves always particles of a different type than α, see (65)–(68). Proceeding again in the same way as in the previous section, we obtain as closed expressions for the n-particle form factors

$$\begin{aligned} &F_{2n}^{\Phi_1^\lambda|n\times\bar{1}1}(\bar{x}_1,x_2,\ldots,\bar{x}_{2n-1},x_{2n}) = (-1)^n F_{2n}^{\Phi_2^{-\lambda}|n\times\bar{2}2}(\bar{x}_1,x_2,\ldots,\bar{x}_{2n-1},x_{2n}) \\ &\quad = F_{2n}^{\Phi_{\bar{1}}^{-\lambda}|n\times\bar{1}1}(\bar{x}_1,x_2,\ldots,\bar{x}_{2n-1},x_{2n}) = (-1)^n F_{2n}^{\Phi_{\bar{2}}^{\lambda}|n\times\bar{2}2}(\bar{x}_1,x_2,\ldots,\bar{x}_{2n-1},x_{2n}) \\ &\quad = i^n 2^{n-1}\sin^n(\pi\lambda)\sigma_n(\bar{x}_1,\ldots,\bar{x}_{2n-1})^{\lambda+\frac{1}{2}}\sigma_n(x_2,\ldots,x_{2n})^{\frac{1}{2}-\lambda}\mathcal{B}_{n,n}. \end{aligned} \tag{82}$$

We may now convince ourselves, that the expressions for $F_{2n}^{\Phi_\alpha^\lambda|n\times\bar{\alpha}\alpha}$ indeed satisfy the consistency equations (2)–(4). The first two equations are rather obvious to check and we will not report this computation here, but the verification of the kinematic residue equation (4) deserves mentioning

$$\begin{aligned} &\mathrm{Res}_{x\to\bar{x}}\, F_{2n+2}^{\Phi_1^\lambda|(n+1)\times\bar{1}1}(-\bar{x},x,\bar{x}_1,x_2,\ldots,\bar{x}_{2n-1},x_{2n}) \\ &\quad = 2^n i^{n+1}\sin^{n+1}(\pi\lambda)\sigma_{n+1}(-x,\bar{x}_1,\ldots,\bar{x}_{2n-1})^{\lambda+\frac{1}{2}}\sigma_{n+1}(x,x_2,\ldots,x_{2n})^{\frac{1}{2}-\lambda} \\ &\qquad \times \mathrm{Res}_{x\to\bar{x}}\,\mathcal{B}_{n+1,n+1} \end{aligned}$$

$$= i\left[1 - \gamma_1^{\Phi_1^\lambda} \prod_{k=1}^{2n} S_{1k}\right] F_{2n}^{\Phi_1^\lambda|n\times\bar{1}1}(\bar{x}_1, \ldots, \bar{x}_{2n-1}, x_{2n}). \tag{83}$$

Recalling the definition of $\mathcal{B}_{n,n}$ of (28), we used $\mathrm{Res}_{x\to\bar{x}}\,\mathcal{B}_{n+1,n+1} = -x^{-1}\mathcal{B}_{n,n}$ and the value for $\gamma_1^{\Phi_1^\lambda}$ from (78). Note the factor $\sin(\pi\lambda)$, which was originally found in [35], and which appears in our presentation in (71) relatively unmotivated, is absolutely crucial for the validity of (83).

Similarly we evaluate the matrix elements of Σ_α^λ

$$\begin{aligned}
&\widetilde{F}_{2n+1}^{\Sigma_1^\lambda|1(n\times\bar{1}1)}(\theta_1, \ldots, \theta_{2n+1}) = (-1)^n \widetilde{F}_{2n+1}^{\Sigma_2^{-\lambda}|2(n\times\bar{2}2)}(\theta_1, \ldots, \theta_{2n+1})\\
&= \widetilde{F}_{2n+1}^{\Sigma_{\bar{1}}^{-\lambda}|1(n\times\bar{1}1)}(\theta_1, \ldots, \theta_{2n+1}) = (-1)^n \widetilde{F}_{2n+1}^{\Sigma_{\bar{2}}^{\lambda}|2(n\times\bar{2}2)}(\theta_1, \ldots, \theta_{2n+1})\\
&= \frac{(2i)^n}{2}\sin^n(\pi\lambda)\frac{\sigma_n(\bar{x}_2, \ldots, \bar{x}_{2n})^{\lambda+\frac{1}{2}}}{\sigma_n(x_1, \ldots, x_{2n+1})^{\lambda-\frac{1}{2}}} \prod_{1\leqslant i<j\leqslant n}(\bar{x}_{2i} - \bar{x}_{2j})\\
&\quad\times \sum_k \frac{i^{k+1}\prod_{j<l;\,j,l\neq k}(x_j - x_l)}{(x_k)^{\frac{1}{2}-\lambda}\prod_{j\neq k}\prod_l(x_j + \bar{x}_l)}.
\end{aligned} \tag{84}$$

However, the expressions of $\widetilde{F}_{2n+1}^{\Sigma_\alpha^\lambda|\alpha(n\times\bar{\alpha}\alpha)}$ only satisfy the consistency equations (2)–(4) for $\lambda = 1/2$. This reflects the fact that $\Sigma_\alpha^\lambda(x)$ is only a local operator for this value of λ, see Eq. (77). Thus, Eqs. (2)–(4) "know" about the locality properties of the operator involved.

As we already commented above, part of the operator content of the Federbush model reduces to the one of the complex fermionic theory. We may check explicitly that the same limit is respected by the form factors

$$\lim_{\lambda\to 1/2} F_{2n}^{\Phi_\alpha^\lambda|n\times\bar{\alpha}\alpha}(\theta_1, \ldots, \theta_{2n}) = F_{2n}^{\mu_\alpha|n\times\bar{\alpha}\alpha}(\theta_1, \ldots, \theta_{2n}), \tag{85}$$

$$\lim_{\lambda\to 1/2} \widetilde{F}_{2n+1}^{\Sigma_\alpha^\lambda|\alpha(n\times\bar{\alpha}\alpha)}(\theta_1, \ldots, \theta_{2n+1}) = F_{2n+1}^{\sigma_\alpha|\alpha(n\times\bar{\alpha}\alpha)}(\theta_1, \ldots, \theta_{2n+1}). \tag{86}$$

Note, however, that since the real fermionic theory cannot be obtained directly from the Federbush model, see also Section 3.1.1, we also do not recover, as we expect, the same expressions for the form factors when the particles are taken to be self-conjugate.

4.2.2. The Federbush fields

Let us now compute the form factors of some fields which occur explicitly in the Federbush model. From the expressions of the previous section the form factors for the Federbush fields follow easily

$$F_{2n+1}^{\Psi_1|(n\times\bar{2}2)1}(\theta_1, \ldots, \theta_{2n+1}) = \sqrt{\pi}\, F_{2n}^{\Phi_2^\lambda|n\times\bar{2}2}(\theta_1, \ldots, \theta_{2n})u_1(\theta_{2n+1}), \tag{87}$$

$$F_{2n+1}^{\Psi_2|(n\times\bar{1}1)2}(\theta_1, \ldots, \theta_{2n+1}) = \sqrt{\pi}\, F_{2n}^{\Phi_1^\lambda|n\times\bar{1}1}(\theta_1, \ldots, \theta_{2n})u_2(\theta_{2n+1}). \tag{88}$$

Recall the definition of the Weyl spinors $u_\alpha(\theta)$ from Eq. (12). It is clear that for each component these fields satisfy the form factor consistency equations. As already mentioned in Section 3.1.2, and as is quite common in the literature, e.g., [37,38], third reference in [6] etc., one may verify that various equations which hold for the operators are also satisfied by the related form factors. However, one should be aware that such relations also hold for the matrix elements $\widetilde{F}$, which do not yet satisfy the consistency equations (2)–(4). Hence, the only conclusion one may draw from such comparisons is a relative consistency amongst the solutions obtained. Such arguments do not serve as a stringent identification of the operators, albeit they give an indication. We illustrate this statement with the following simple computation. Let us take the fields as defined in (70) and evaluate directly by Wick contracting

$$\widetilde{F}_2^{\partial_0\phi_\alpha|\bar{\alpha}\alpha}(\theta,\tilde{\theta}) = \sqrt{\pi}\,\widetilde{F}_2^{J_\alpha^1|\bar{\alpha}\alpha}(\theta,\tilde{\theta}) = -i\pi^{\frac{3}{2}} m_\alpha \cosh\frac{\theta+\tilde{\theta}}{2}, \quad \alpha = 1,2, \tag{89}$$

$$\widetilde{F}_2^{\partial_1\phi_\alpha|\bar{\alpha}\alpha}(\theta,\tilde{\theta}) = -\sqrt{\pi}\,\widetilde{F}_2^{J_\alpha^0|\bar{\alpha}\alpha}(\theta,\tilde{\theta}) = i\pi^{\frac{3}{2}} m_\alpha \sinh\frac{\theta+\tilde{\theta}}{2}, \quad \alpha = 1,2. \tag{90}$$

This confirms precisely the conservation equations (52) on the level of the matrix elements. However, it is also easy to see that the expressions (89) and (90) are not yet solutions of the form factor consistency equations (2)–(4). In principle, these equations together with the Dirac equation already ensure that the Ψ_α are solutions of the equations of motion (50). Nonetheless, it is instructive to verify (50) explicitly. Using still the representation (70), we compute

$$\widetilde{F}_3^{\varepsilon_{\mu\nu}J_2^\nu\gamma^\mu\Psi_1|\bar{2}21}(\theta_1,\theta_2,\theta_3) = i\pi^{\frac{3}{2}} m_2 u_1(\theta_1+\theta_2-\theta_3), \tag{91}$$

$$\widetilde{F}_3^{\gamma^\mu\partial_\mu\Psi_1|\bar{2}21}(\theta_1,\theta_2,\theta_3) = \frac{i\pi^{\frac{5}{2}}\lambda(m_2u_1(\theta_{13}+\theta_1)+m_2u_1(\theta_{23}+\theta_2)+m_1u_1(\theta_3))}{\cosh\frac{1}{2}\theta_{12}}, \tag{92}$$

$$\widetilde{F}_3^{\Psi_1|\bar{2}21}(\theta_1,\theta_2,\theta_3) = \frac{i\pi^{\frac{5}{2}}\lambda}{\cosh\frac{1}{2}\theta_{12}} u_1(\theta_3). \tag{93}$$

Assembling these expressions, we confirm directly the validity of (50) at the level of the three particle matrix elements. We expect of course this property also to holds for higher orders. It is easy to check that (91)–(93) do not constitute solutions of the Eqs. (2)–(4), in particular, (87) does not reduce to (93). Thus on one hand we see that formal operator equations do not serve as a conclusive means of operator identification and we therefore need alternative arguments such as the ultraviolet limit in Section 5 etc. On the other hand this underlines further the need for the introduction of the field $\Omega_\alpha^\lambda(x)$.

4.2.3. The energy–momentum tensor

The energy–momentum tensor for the Federbush model has been computed in [39]. Its evaluation involved a small subtlety, since the one obtained directly from the Lagrangian does not lead to the correct Poincaré generators, such as (47). This could be fixed in the usual way by exploiting the ambiguity in the definition. Essential for our purposes is once again the trace, which is

$$T^\mu{}_\mu = 2im_1{:}\overline{\Psi}_1\Psi_1{:} + 2im_2{:}\overline{\Psi}_2\Psi_2{:}. \tag{94}$$

Using the representation (51) for the Federbush fields, we compute the only nonvanishing form factor for ${T^\mu}_\mu$ to

$$F_2^{{T^\mu}_\mu|\bar{\alpha}\alpha}(\theta,\tilde{\theta}) = F_2^{{T^\mu}_\mu|\alpha\bar{\alpha}}(\theta,\tilde{\theta}) = -2\pi i m_\alpha^2 \sinh\frac{\theta-\tilde{\theta}}{2}. \tag{95}$$

This means the function is the same as the one for the complex free fermion.

4.3. *Momentum space cluster properties*

As a consequence of Weinberg's power counting theorem one has also a further property of form factors which involves the structure of the operators themselves, namely, the momentum space cluster property, see, e.g., [1] some reasoning on this. It serves on one hand as a consistency check for possible solutions of (2)–(4) and on the other as a construction principle for new solutions, e.g., [36]. It states that whenever some of the rapidities, say κ, are shifted to plus or minus infinity, the n-particle form factor related to a local operator $\mathcal{O}$ factorizes into a κ and an $(n-\kappa)$-particle form factor which are possibly related to different types of operators $\mathcal{O}'$ and $\mathcal{O}''$. Introducing the translation operator T_a^ϑ which acts on a function of n variables as

$$T_a^\vartheta f(\theta_1,\ldots,\theta_a,\ldots,\theta_n) \mapsto f(\theta_1,\ldots,\theta_a+\vartheta,\ldots,\theta_n) \tag{96}$$

and the operators

$$\overline{\mathcal{T}}_{a,b}^{\pm} = \lim_{\vartheta\to\infty}\prod_{p=a}^{b} T_{2p-1}^{\pm\vartheta} \quad \text{and} \quad \mathcal{T}_{a,b}^{\pm} = \lim_{\vartheta\to\infty}\prod_{p=a}^{b} T_{2p}^{\pm\vartheta}, \tag{97}$$

the statement of momentum space cluster decomposition reads

$$\begin{aligned}
&\overline{\mathcal{T}}_{a,\kappa}^{\pm}\mathcal{T}_{a,\kappa}^{\pm} F_n^{\mathcal{O}}(\theta_1\cdots\theta_n)\\
&\quad\sim F_{2(\kappa-a+1)}^{\mathcal{O}'}(\theta_{2a-1}\cdots\theta_{2\kappa}) F_{n-2(\kappa-a+1)}^{\mathcal{O}''}(\theta_1\cdots\theta_{2a-2},\theta_{2\kappa+1}\cdots\theta_n).
\end{aligned} \tag{98}$$

Of course, we could have defined the product of $\overline{\mathcal{T}}_{a,\kappa}^{\pm}\mathcal{T}_{a,\kappa}^{\pm}$ to be just one operator, but it will be convenient for us to distinguish the shifts in even and odd positions of the particles. Let us now see the effect of the action of these operators on the various functions which build up our form factor solutions, see (27), (33) and (82). We compute

$$\overline{\mathcal{T}}_{1,\kappa}^{\pm}\mathcal{T}_{1,\zeta}^{\pm}\left[\frac{\sigma_n(\bar{x}_1\cdots\bar{x}_{2n-1})^{\lambda+\frac{1}{2}}}{\sigma_m(x_2\cdots x_{2m})^{\lambda-\frac{1}{2}}}\right] \sim e^{\pm\lambda\vartheta(\kappa-\zeta)\pm\vartheta\frac{(\kappa+\zeta)}{2}}\left[\frac{\sigma_n(\bar{x}_1\cdots\bar{x}_{2n-1})^{\lambda+\frac{1}{2}}}{\sigma_m(x_2\cdots x_{2m})^{\lambda-\frac{1}{2}}}\right], \tag{99}$$

and

$$\overline{\mathcal{T}}_{1,\kappa}^{\pm}\mathcal{T}_{1,\zeta}^{\pm}\mathcal{B}_{n,m} \sim \mathcal{B}_{\kappa,\varsigma}\mathcal{B}_{n-\kappa,m-\varsigma}\begin{cases} e^{\vartheta(\zeta-\kappa)[\frac{\kappa-\zeta}{2}-n+m]-\vartheta\frac{(\kappa+\zeta)}{2}}\left[\frac{\sigma_\kappa(\bar{x}_1\cdots\bar{x}_{2\kappa-1})}{\sigma_\varsigma(x_2\cdots x_{2\varsigma})}\right]^{n-m+\varsigma-\kappa}, \\ e^{-\vartheta\frac{(\kappa-\zeta)^2}{2}+\vartheta\frac{(\kappa+\zeta)}{2}}\left[\frac{\sigma_{n-\kappa}(\bar{x}_{2\kappa+1}\cdots\bar{x}_{2n-1})}{\sigma_{m-\varsigma}(x_{2\varsigma+2}\cdots x_{2m})}\right]^{\kappa-\varsigma}. \end{cases} \tag{100}$$

In order not to overload our symbols, we have slightly abused here the notation. Whereas in (28) the $x_i,\bar{x}_j$-dependence of $\mathcal{B}_{n,m}$ always start at $i,j=1$, in (100) the dependence of

$\mathcal{B}_{n-\kappa,m-\varsigma}$ for the minus shift is the same as in the corresponding factor for the symmetric polynomials. Besides the explicit functional dependence on the r.h.s. of (99) and (100) it is instructive to consider at first the leading order behaviour

$$\overline{\mathcal{T}}^{\pm}_{1,\kappa}\mathcal{T}^{\pm}_{1,\zeta}\left[\frac{\sigma_n(\bar{x}_1\cdots\bar{x}_{2n-1})^{\lambda+\frac{1}{2}}}{\sigma_m(x_2\cdots x_{2m})^{\lambda-\frac{1}{2}}}\right]\mathcal{B}_{n,m}\sim\begin{cases}e^{\vartheta(\zeta-\kappa)[\frac{\kappa-\zeta}{2}-n+m-\lambda]},\\ e^{\vartheta(\zeta-\kappa)[\frac{\kappa-\zeta}{2}+\lambda]}.\end{cases}\tag{101}$$

From this we see directly that in general the final expression will tend to zero, unless $\zeta=\kappa$, $|\zeta-\kappa|=2\lambda$ or $|\zeta-\kappa\pm 2|=2\lambda$, by noting that our solutions only allow $|n-m|=0,1$. So, let us now collect the functional dependences in Eqs. (99) and (100) and see how our form factor solutions combine under clustering to new form factors. We compute

$$\overline{\mathcal{T}}^{\pm}_{1,\kappa}\mathcal{T}^{\pm}_{1,\kappa}F_{2n}^{\Phi^{\lambda}_{\alpha}|n\times\bar{\alpha}\alpha}\sim F_{2\kappa}^{\Phi^{\lambda}_{\alpha}|\kappa\times\bar{\alpha}\alpha}(\theta_1\cdots\theta_{2\kappa})F_{2(n-\kappa)}^{\Phi^{\lambda}_{\alpha}|(n-\kappa)\times\bar{\alpha}\alpha}(\theta_{2\kappa+1}\cdots\theta_{2n}),\tag{102}$$

$$\overline{\mathcal{T}}^{\pm}_{1,\kappa}\mathcal{T}^{\pm}_{1,\kappa}F_{2n}^{\Phi^{\lambda}_{\bar{\alpha}}|n\times\bar{\alpha}\alpha}\sim F_{2\kappa}^{\Phi^{\lambda}_{\bar{\alpha}}|\kappa\times\bar{\alpha}\alpha}(\theta_1\cdots\theta_{2\kappa})F_{2(n-\kappa)}^{\Phi^{\lambda}_{\bar{\alpha}}|(n-\kappa)\times\bar{\alpha}\alpha}(\theta_{2\kappa+1}\cdots\theta_{2n}),\tag{103}$$

$$\overline{\mathcal{T}}^{\pm}_{1,\kappa+1}\mathcal{T}^{\pm}_{1,\kappa}F_{2n}^{\Phi^{\pm\frac{1}{2}}_{\alpha}|n\times\bar{\alpha}\alpha}\sim F_{2\kappa+1}^{\sigma_{\bar{\alpha}}|(\kappa\times\bar{\alpha}\alpha)\bar{\alpha}}(\theta_1\cdots\theta_{2\kappa+1})F_{2(n-\kappa)-1}^{\sigma_{\alpha}|[(n-\kappa)\times\bar{\alpha}\alpha]\alpha}(\theta_{2\kappa+2}\cdots\theta_{2n}),\tag{104}$$

$$\overline{\mathcal{T}}^{\pm}_{1,\kappa}\mathcal{T}^{\pm}_{1,\kappa+1}F_{2n}^{\Phi^{\mp\frac{1}{2}}_{\alpha}|n\times\bar{\alpha}\alpha}\sim F_{2\kappa+1}^{\sigma_{\alpha}|(\kappa\times\bar{\alpha}\alpha)\alpha}(\theta_1\cdots\theta_{2\kappa+1})F_{2(n-\kappa)-1}^{\sigma_{\bar{\alpha}}|[(n-\kappa)\times\bar{\alpha}\alpha]\bar{\alpha}}(\theta_{2\kappa+2}\cdots\theta_{2n}),\tag{105}$$

$$\overline{\mathcal{T}}^{+}_{1,\kappa}\mathcal{T}^{+}_{1,\kappa}F_{2n+1}^{\sigma_{\alpha}|(n\times\bar{\alpha}\alpha)\alpha}\sim F_{2\kappa}^{\mu_{\bar{\alpha}}|\kappa\times\bar{\alpha}\alpha}(\theta_1\cdots\theta_{2\kappa})F_{2(n-\kappa)+1}^{\sigma_{\alpha}|[(n-\kappa)\times\bar{\alpha}\alpha]\alpha}(\theta_{2\kappa+1}\cdots\theta_{2n+1}),\tag{106}$$

$$\overline{\mathcal{T}}^{-}_{1,\kappa}\mathcal{T}^{-}_{1,\kappa}F_{2n+1}^{\sigma_{\alpha}|(n\times\bar{\alpha}\alpha)\alpha}\sim F_{2\kappa}^{\mu_{\alpha}|\kappa\times\bar{\alpha}\alpha}(\theta_1\cdots\theta_{2\kappa})F_{2(n-\kappa)+1}^{\sigma_{\alpha}|[(n-\kappa)\times\bar{\alpha}\alpha]\alpha}(\theta_{2\kappa+1}\cdots\theta_{2n+1}),\tag{107}$$

$$\overline{\mathcal{T}}^{+}_{1,\kappa}\mathcal{T}^{+}_{1,\kappa+1}F_{2n+1}^{\sigma_{\alpha}|(n\times\bar{\alpha}\alpha)\alpha}\sim F_{2\kappa+1}^{\sigma_{\alpha}|(\kappa\times\bar{\alpha}\alpha)\alpha}(\theta_1\cdots\theta_{2\kappa+1})F_{2(n-\kappa)}^{\mu_{\alpha}|(n-\kappa)\times\bar{\alpha}\alpha}(\theta_{2\kappa+2}\cdots\theta_{2n+1}),\tag{108}$$

$$\overline{\mathcal{T}}^{-}_{1,\kappa}\mathcal{T}^{-}_{1,\kappa+1}F_{2n+1}^{\sigma_{\alpha}|(n\times\bar{\alpha}\alpha)\alpha}\sim F_{2\kappa+1}^{\sigma_{\alpha}|(\kappa\times\bar{\alpha}\alpha)\alpha}(\theta_1\cdots\theta_{2\kappa+1})F_{2(n-\kappa)}^{\mu_{\bar{\alpha}}|(n-\kappa)\times\bar{\alpha}\alpha}(\theta_{2\kappa+2}\cdots\theta_{2n+1}),\tag{109}$$

$$\overline{\mathcal{T}}^{+}_{1,\kappa}\mathcal{T}^{+}_{1,\kappa}F_{2n+1}^{\sigma_{\bar{\alpha}}|(n\times\bar{\alpha}\alpha)\bar{\alpha}}\sim F_{2\kappa}^{\mu_{\alpha}|\kappa\times\bar{\alpha}\alpha}(\theta_1\cdots\theta_{2\kappa})F_{2(n-\kappa)+1}^{\sigma_{\bar{\alpha}}|[(n-\kappa)\times\bar{\alpha}\alpha]\bar{\alpha}}(\theta_{2\kappa+1}\cdots\theta_{2n+1}),\tag{110}$$

$$\overline{\mathcal{T}}^{-}_{1,\kappa}\mathcal{T}^{-}_{1,\kappa}F_{2n+1}^{\sigma_{\bar{\alpha}}|(n\times\bar{\alpha}\alpha)\bar{\alpha}}\sim F_{2\kappa}^{\mu_{\bar{\alpha}}|\kappa\times\bar{\alpha}\alpha}(\theta_1\cdots\theta_{2\kappa})F_{2(n-\kappa)+1}^{\sigma_{\bar{\alpha}}|[(n-\kappa)\times\bar{\alpha}\alpha]\bar{\alpha}}(\theta_{2\kappa+1}\cdots\theta_{2n+1}),\tag{111}$$

$$\overline{\mathcal{T}}^{+}_{1,\kappa+1}\mathcal{T}^{+}_{1,\kappa}F_{2n+1}^{\sigma_{\bar{\alpha}}|(n\times\bar{\alpha}\alpha)\bar{\alpha}}\sim F_{2\kappa+1}^{\sigma_{\bar{\alpha}}|(\kappa\times\bar{\alpha}\alpha)\bar{\alpha}}(\theta_1\cdots\theta_{2\kappa+1})F_{2(n-\kappa)}^{\mu_{\bar{\alpha}}|(n-\kappa)\times\bar{\alpha}\alpha}(\theta_{2\kappa+2}\cdots\theta_{2n+1}),\tag{112}$$

$$\overline{\mathcal{T}}_{1,\kappa+1}^{-}\mathcal{T}_{1,\kappa}^{-}F_{2n+1}^{\sigma_{\bar{\alpha}}|(n\times\bar{\alpha}\alpha)\bar{\alpha}} \sim F_{2\kappa+1}^{\sigma_{\bar{\alpha}}|(\kappa\times\bar{\alpha}\alpha)\bar{\alpha}}(\theta_1\cdots\theta_{2\kappa+1})F_{2(n-\kappa)}^{\mu_\alpha|(n-\kappa)\times\bar{\alpha}\alpha}(\theta_{2\kappa+2}\cdots\theta_{2n+1}). \tag{113}$$

Thus, omitting the shift operators we have formally the following decomposition of the operators

$$\Phi_\alpha^\lambda \longrightarrow \Phi_\alpha^\lambda \times \Phi_\alpha^\lambda, \qquad \sigma_\alpha \longrightarrow \begin{cases} \mu_\alpha \times \sigma_\alpha, \\ \mu_{\bar{\alpha}} \times \sigma_\alpha, \end{cases} \qquad \mu_\alpha \longrightarrow \begin{cases} \mu_\alpha \times \mu_\alpha, \\ \sigma_\alpha \times \sigma_{\bar{\alpha}}, \end{cases} \tag{114}$$

together with the equations for $\alpha \leftrightarrows \bar{\alpha}$. This means the stated operator content closes consistently under the action of the cluster decomposition operators.

5. Lie algebraically coupled Federbush models

The Federbush model as investigated in the previous section only contains two types of particles. In this section we propose a new Lagrangian, which admits a much larger particle content. The theories are not yet as complex as the HSG-models, but they can also be obtained from them in a certain limit such that they will always constitute a benchmark for these class theories. Form factors related to these models may be computed similarly as in the previous section.

Let us consider $\ell \times \tilde{\ell}$-real (Majorana) free fermions $\psi_{a,j}(x)$, now labeled by two quantum numbers $1 \leqslant a \leqslant \ell$, $1 \leqslant j \leqslant \tilde{\ell}$ and described by the Dirac Lagrangian density $\mathcal{L}_{\mathrm{FF}}$. We perturb this system with a bilinear term in the vector currents $J_{a,j}^\mu = \overline{\Psi}_{a,j}\gamma^\mu\Psi_{a,j}$

$$\mathcal{L}_{\mathrm{CF}} = \sum_{a=1}^{\ell}\sum_{j=1}^{\tilde{\ell}} \overline{\Psi}_{a,j}\big(i\gamma^\mu\partial_\mu - m_{a,j}\big)\Psi_{a,j} - \frac{1}{2}\pi\varepsilon_{\mu\nu}\sum_{a,b=1}^{\ell}\sum_{j,k=1}^{\tilde{\ell}} J_{a,j}^\mu J_{b,k}^\nu \Lambda_{ab}^{jk}, \tag{115}$$

and denote the new fields in $\mathcal{L}_{\mathrm{CF}}$ by $\Psi_{a,j}$. Furthermore, we introduced $(\ell^2 \times \tilde{\ell}^2)$-dimensional coupling constant dependent matrix Λ_{ab}^{jk}, whose further properties we leave unspecified at this stage. As in the usual Federbush model, the effect of the presence of the Levi-Civita pseudotensor ε is that the theory described by $\mathcal{L}_{\mathrm{CF}}$ is not parity invariant. Thus $\mathcal{L}_{\mathrm{CF}}$ may be viewed as a system of coupled Federbush models [25].

The formal equations of motion associated to $\mathcal{L}_{\mathrm{CF}}$ are easily derived as

$$\big(i\gamma^\mu\partial_\mu - m_{a,j}\big)\Psi_{a,j} = \pi\varepsilon_{\mu\nu}\gamma^\mu\sum_{b=1}^{\ell}\sum_{k=1}^{\tilde{\ell}}\Lambda_{ab}^{jk}J_{b,k}^\mu\Psi_{a,j}. \tag{116}$$

The solutions to these equations can be constructed in close analogy to the ones of the Federbush model. The fields

$$\Psi_{a,j} = \vdots\exp\left(\sqrt{\pi}\,i\sum_{b=1}^{\ell}\sum_{k=1}^{\tilde{\ell}}\Lambda_{ab}^{jk}\phi_{b,k}\right)\vdots\,\psi_{a,j} = \Phi_{a,j}^\lambda\psi_{a,j} \tag{117}$$

solve the equations of motion (116) with the additional assumption that the bosonic fields $\phi_{a,j}$ constitute potentials for axial vector currents

$$\frac{1}{\sqrt{\pi}}\partial_\mu \phi_{a,j} = \varepsilon_{\nu\mu} J^\nu_{a,j} = \bar{\psi}_{a,j}\gamma_\mu\gamma^5\psi_{a,j}, \qquad \Lambda^{jk}_{ab} \neq 0, \quad \forall b,k. \tag{118}$$

As in the previous section, we used here once again the triple normal ordering in Eq. (117). It needs further computations, similar to the ones for the Federbush model, to make it rigorous that also in this context the triple ordering can be associated to a standard Wick normal ordering. Nonetheless, it appears natural to expect that this can be generalized analogously and we take this here as an assumption.

Accepting this, we can now compute various equal time exchange relations with $1 \leqslant a, b \leqslant \ell$, $1 \leqslant j, k \leqslant \tilde{\ell}$

$$\left[\phi_{a,j}(x), \phi_{b,k}(y)\right] = \left[\Phi_{a,j}(x), \Phi_{b,k}(y)\right] = 0, \tag{119}$$

$$\left[\phi_{a,j}(x), \Phi_{b,k}(y)\right] = \left\{\psi_{a,j}(x), \psi_{b,k}(y)\right\} = 0, \tag{120}$$

$$\left[\psi_{a,j}(x), \phi_{b,k}(y)\right] = \sqrt{\pi}\,\delta_{a,b}\delta_{j,k}\Theta\left(x^1 - y^1\right)\psi_{a,j}(x), \tag{121}$$

$$\psi_{a,j}(x)\Phi^\lambda_{b,k}(y) = \Phi^\lambda_{b,k}(y)\psi_{a,j}(x)e^{-i\pi\Lambda^{jk}_{ab}\Theta(x^1-y^1)}, \tag{122}$$

$$-\psi_{a,j}(x)\Psi_{b,k}(y) = \Psi_{b,k}(y)\psi_{a,j}(x)e^{-i\pi\Lambda^{jk}_{ab}\Theta(x^1-y^1)}, \tag{123}$$

$$\Psi_{a,j}(x)\Phi^\lambda_{b,k}(y) = \Phi^\lambda_{b,k}(y)\Psi_{a,j}(x)e^{-i\pi\Lambda^{jk}_{ab}\Theta(x^1-y^1)}, \tag{124}$$

$$-\Psi_{a,j}(x)\Psi_{b,k}(y) = \Psi_{b,k}(y)\Psi_{a,j}(x)e^{-i\pi\Lambda^{jk}_{ab}}. \tag{125}$$

Eqs. (119) and (120) are again clear since $\psi_{a,j}$ and $\phi_{a,j}$ are free fermions and bosons, respectively. Eq. (121) is compatible with (118) and the remaining equations are simply consequences of (119)–(121). With the help of these equations we compute directly the scattering matrix. We will be slightly casual here about complete rigour and do not worry with test functions and smeared out operators. Noting that $\phi_{a,j}\,|0\rangle = 0$, we obtain from

$$\begin{aligned}
&\lim_{t\to-\infty} \Psi_{a,j}\Psi_{b,k}|0\rangle = \check{S}^{jk}_{ab}\psi_{a,j}\psi_{b,k}|0\rangle, \\
&\lim_{t\to+\infty} \Psi_{a,j}\Psi_{b,k}|0\rangle = \widehat{S}^{jk}_{ab}\psi_{b,k}\psi_{a,j}|0\rangle
\end{aligned} \tag{126}$$

the S-matrix

$$S^{jk}_{ab} = \left(\widehat{S}^{jk}_{ab}\right)^{-1}\check{S}^{jk}_{ab} = -e^{i\pi\Lambda^{jk}_{ab}}. \tag{127}$$

Let us now see whether (127) is consistent in the usual sense, i.e., that it passes all the tests of consistency or if the latter put some constraints on the possible values for the coupling constant dependent matrix Λ^{jk}_{ab}. We demand the usual crossing and unitarity relations (61), which means we should have

$$\Lambda^{jk}_{ab} = -\Lambda^{kj}_{ba} + 2\mathbb{Z} \quad \text{and} \quad \Lambda^{jk}_{ab} = \Lambda^{\bar{k}j}_{\bar{b}a} + 2\mathbb{Z}. \tag{128}$$

We will now provide some concrete solutions to (128) and, therefore, (127).

5.1. HSG-type solutions

Let us take

$$\Lambda_{ab}^{jk} = 2\lambda_{ab}\varepsilon_{jk}\tilde{I}_{jk}K_{a\bar{b}}^{-1}, \tag{129}$$

where K denotes the Cartan matrix of $SU(N)$ and $\tilde{I}$ the incidence matrix of a simply laced Lie algebra, which we refer to as $\tilde{g}$. The λ_{ab} are ℓ^2 coupling constants, which are, however, not entirely independent of each other. For instance we assume $\lambda_{ab} = \lambda_{ba}$. Furthermore, we characterise the antiparticle exclusively by the first quantum number, i.e., $\overline{(a,i)} = (\bar{a}, i)$, where the particle $\bar{a}$ may be constructed from a by the automorphism which leaves the associated Dynkin diagram invariant. In the case of $SU(N)$, we simply have $\bar{a} = \ell + 1 - a$. It is clear that (129) satisfies the first relation in (128), whereas the second relation introduces further constraints on the λ's. To be more concrete we specify now (127) for some special choices of N and the Lie algebra $\tilde{g}$.

5.2. The Federbush model

Considering now the case $SU(3)_3$ with $\lambda_{11} = \lambda_{22} = -2\lambda_{12} = -2\lambda_{21} = \lambda$, we obtain the scattering matrix of the Federbush model S^{FB} as defined in (64), where we now used the ordering $\{(1,1), (2,1), (1,2), (2,2)\}$. In comparison with the previous section, one should notice, that we have now realised this model in terms real fermions rather than complex ones.

5.3. $\tilde{g}_6$

To illustrate the formulae (128) and (127) a bit more, let us consider a slighly more complex model, namely, $\tilde{g}_6$. When specifying the quantities in (129) to these algebras, we obtain

$$S^{ij} = -\begin{pmatrix} e^{2\pi i\lambda\varepsilon_{ij}I_{ij}} & e^{2\pi i\lambda'\varepsilon_{ij}I_{ij}} & 1 & e^{-2\pi i\lambda'\varepsilon_{ij}I_{ij}} & e^{-2\pi i\lambda\varepsilon_{ij}I_{ij}} \\ e^{-2\pi i\lambda'\varepsilon_{ij}I_{ij}} & e^{2\pi i\lambda''\varepsilon_{ij}I_{ij}} & 1 & e^{-2\pi i\lambda''\varepsilon_{ij}I_{ij}} & e^{2\pi i\lambda'\varepsilon_{ij}I_{ij}} \\ 1 & 1 & 1 & 1 & 1 \\ e^{2\pi i\lambda'\varepsilon_{ij}I_{ij}} & e^{-2\pi i\lambda''\varepsilon_{ij}I_{ij}} & 1 & e^{2\pi i\lambda''\varepsilon_{ij}I_{ij}} & e^{-2\pi i\lambda'\varepsilon_{ij}I_{ij}} \\ e^{-2\pi i\lambda\varepsilon_{ij}I_{ij}} & e^{-2\pi i\lambda'\varepsilon_{ij}I_{ij}} & 1 & e^{2\pi i\lambda'\varepsilon_{ij}I_{ij}} & e^{2\pi i\lambda\varepsilon_{ij}I_{ij}} \end{pmatrix}, \tag{130}$$

where the rows and columns are ordered as $1, 2, \ldots, 5$. In this case, we have three independent coupling constants λ, λ' and λ''.

5.4. The HSG-limit

From the specific example in (64), we expect that the HSG-models are in general closely related to (127) with (129). Indeed, taking $\lambda_{ab} = 1$ for $1 \leqslant a, b \leqslant \ell$, we obtain

$$S_{ab}^{jk} = -e^{2\pi i\varepsilon_{jk}I_{jk}K_{a\bar{b}}^{-1}}, \tag{131}$$

which clearly satisfies the first relation in (128), whereas for the second relation, we simply have to recall the well-known fact that $\left(K^{-1}_{SU(N)}\right)_{ab} = \min(a,b) - ab/N$. Comparing with the expression

$$\lim_{\theta\to\infty}\left[S^{jk\mathrm{HSG}}_{ab}(\theta)S^{jk\mathrm{HSG}}_{ab}(-\theta)\right] = e^{2\pi i (K^{SU(N)}_{\bar{a}b})^{-1} I}, \tag{132}$$

we note that these solutions coincide. This means when one eventually solves the HSG-models, one can always take the limit to the corresponding quantities of $\mathcal{L}_{\mathrm{CF}}$ for consistency checks.

6. The ultraviolet limit

When having found a solution to the form factor consistency equations, with the factor of local commutativity and the scattering matrix as the only input, one normally does not know which operator this particular solution corresponds to. Of course in the present situation we are in a better position, since we are already working with an explicit representation for the operators. Nonetheless, in Section 4.2.2, we saw that even this can still lead to wrong assignments and it is desirable to have more information. By calling the operators Φ^{λ}_{α}, μ, and $\Sigma^{\lambda}_{\alpha}$, σ, disorder and order operators, respectively, we have already borrowed the terminology from the underlying conformal field theory. In order to make this correspondence more manifest one may carry out explicitly the ultraviolet limit. The ultraviolet Virasoro central charge of the theory itself can be computed from the knowledge of the form factors of the trace of the energy–momentum tensor [23] by means of the expansion

$$c_{\mathrm{uv}} = \sum_{n=1}^{\infty}\sum_{\mu_1\cdots\mu_n}\frac{9}{n!(2\pi)^n}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{d\theta_1\cdots d\theta_n}{\left(\sum_{i=1}^{n} m_{\mu_i}\cosh\theta_i\right)^4}\left|F_n^{T^{\mu}{}_{\mu}|\mu_1\ldots\mu_n}(\theta_1,\ldots,\theta_n)\right|^2. \tag{133}$$

In a similar way one may compute the scaling dimension of the operator $\mathcal{O}$ from the knowledge of its n-particle form factors [24]

$$\begin{aligned}\Delta^{\mathcal{O}}_{\mathrm{uv}} = -\frac{1}{2\langle\mathcal{O}\rangle}\sum_{n=1}^{\infty}\sum_{\mu_1\cdots\mu_n}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}&\frac{d\theta_1\cdots d\theta_n}{n!(2\pi)^n\left(\sum_{i=1}^{n} m_{\mu_i}\cosh\theta_i\right)^2}\\ &\times F_n^{T^{\mu}{}_{\mu}|\mu_1\cdots\mu_n}(\theta_1,\ldots,\theta_n)\left(F_n^{\mathcal{O}|\mu_1\ldots\mu_n}(\theta_1,\ldots,\theta_n)\right)^*.\end{aligned} \tag{134}$$

In general the expressions (133) and (134) yield the difference between the corresponding infrared and ultraviolet values, but we assumed here already that the theory is purely massive such that the infrared contribution vanishes. Let us now evaluate these formulae.

6.1. The complex fermion

Since for the free fermion one only has to sum up to the two particle contribution, the infinite sum (133) and (134) terminate and can be evaluated even analytically. For the case

$N=2$ we obtain

$$c_{\mathrm{uv}}=2 \quad \text{and} \quad \Delta_{\mathrm{uv}}^{\mu_\alpha}=\Delta_{\mathrm{uv}}^{\mu_{\bar\alpha}}=\frac{1}{16}. \tag{135}$$

The scaling dimensions of σ_α and $\sigma_{\bar\alpha}$, which are expected to coincide with (135), cannot be computed from (134), since it involves an odd number of particles.

6.2. The Federbush model

We may proceed similarly for the Federbush model. In the ultraviolet limit it obviously corresponds to two complex free fermions and we there expect to obtain

$$c_{\mathrm{uv}}=2. \tag{136}$$

Indeed using (95), the computation is identical to the one carried out in the previous section. Note, that this value of 2 coincides with the ultraviolet central charge of the $SU(3)_3$-HSG model. This is, however, also not entirely surprising by recalling the identification (64). The corresponding thermodynamic Bethe ansatz equations will be identical to for free fermions. More striking is the result of the evaluation of (134), which yields with (95) and (81)

$$\Delta_{\mathrm{uv}}^{\Phi_\alpha^\lambda}=\Delta_{\mathrm{uv}}^{\Phi_{\bar\alpha}^\lambda}=\frac{\lambda^2}{4}. \tag{137}$$

Note, that $\Delta_{\mathrm{uv}}^{\Phi_\alpha^{1/2}}=\Delta_{\mathrm{uv}}^{\Phi_{\bar\alpha}^{1/2}}=1/16$, which is once again the limit to the complex free fermion. Yet more support for the relation between the $SU(3)_3$-HSG model and the Federbush model comes from the analysis of $\lambda=2/3$, which corresponds to the $SU(3)_3$-HSG value $\tau=1/3$ (see (63)). In that case we obtain from (137) $\Delta_{\mathrm{uv}}^{\Phi_\alpha^{2/3}}=\Delta_{\mathrm{uv}}^{\Phi_{\bar\alpha}^{2/3}}=1/9$. We now compare with the general formula for the scaling dimensions of the $SU(3)_3$-HSG model

$$\Delta(\Lambda,w)=\frac{(\Lambda\cdot(\Lambda+2\rho))}{12}-\frac{(w\cdot w)}{6}, \tag{138}$$

where Λ is a highest weight vector of level smaller or equal 3, w the corresponding lower weights and ρ the Weyl vector. We are specially interested in the field corresponding to $\Delta(\lambda_1,\lambda_1)$ with λ_1 being a fundamental weight, since this field was previously observed [36] to correspond to the disorder operator. Indeed, we find that

$$\Delta(\lambda_1,\lambda_1)=\Delta_{\mathrm{uv}}^{\Phi_\alpha^{2/3}}=\Delta_{\mathrm{uv}}^{\Phi_{\bar\alpha}^{2/3}}. \tag{139}$$

Thus precisely at the value of the coupling constant of the Federbush model at which the $SU(3)_3$-HSG S-matrix reduces in the limit (64) to the S^{FB}, the operator content of the two models overlaps.

7. Conclusions

We summarize our main results supplemented by a short characterization:

We computed explicitly closed formulae for the n-particle form factors of the complex free fermion and the Federbush model related to various operators.

We carried out this computations in two alternative ways: On the one hand, we represent explicitly the field content (14) as well as the particle creation operators (8) in terms of fermionic Fock operators (9) and computed thereafter directly the corresponding matrix elements. On the other hand we verified that these expressions satisfy the form factor consistency equations only when the operators under consideration are mutually local, i.e., satisfying (5). This can already be seen for the free fermion, for which we could have also computed the matrix element of the field $\Phi_\alpha^\lambda(x)$. In that context one observes that only for $\lambda = 1/2$ the resulting function $\widetilde{F}$ solves the consistency equations (2)–(4). We observed a similar phenomenon in the Federbush model. Whereas the matrix elements of the field $\Sigma_\alpha^\lambda(x)$ can be computed in a closed form for generic values of λ, they become only meaningful form factors for $\lambda = 1/2$, that is when the field becomes local. This means it is crucial that the consistency equations contain the factor of local commutativity $\gamma_\mu^{\mathcal{O}}$ as defined in (5), which we computed from first principles with the help of (53)–(59).

Our solutions turned out to decompose consistently under the momentum space cluster property. This computations constitute next to the ones in [36] the only concrete examples of nonselfclustering, i.e., $\mathcal{O} \to \mathcal{O}' \times \mathcal{O}''$ in the sense of (114).

Further support for the identification of the solutions with a specific operator was given by an analysis of the ultraviolet limit.

We demonstrated how the scattering matrix of the Federbush model can be obtained as a limit of the $SU(3)_3$-HSG scattering matrix. This "correspondence" also holds for the central charge, which equals 2 in both cases, and the scaling dimension of the disorder operator at a certain value of the coupling constant.

We proposed a Lie algebraic generalization of the Federbush models, by suggesting a new type of Lagrangian. We evaluate from first principles the related scattering matrices, which can also be obtained in a certain limit from the HSG-models.

We expect that the construction of form factors by means of free fermionic Fock fields can be extended to other models.

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