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Form factors of the homogeneous sine-Gordon models

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Abstract

We provide general determinant formulae for all n-particle form factors related to the trace of the energy momentum tensor and the analogue of the order and disorder operator in the $SU(3)_2$-homogeneous sine-Gordon model. We employ the form factors related to the trace of the energy momentum tensor in the application of the c-theorem and find perfect agreement with the physical picture recently obtained by means of the thermodynamic Bethe ansatz. For finite resonance parameter we recover the expected WZNW-coset central charge and for infinite resonance parameter the theory decouples into two free fermions.

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1. Introduction

In [1] a certain physical picture for the quantum field theory of the Homogeneous Sine-Gordon models (HSG) [2] was extracted from a thermodynamic Bethe ansatz analysis. The central aim of this manuscript is to inspect the picture for consistency by means of the form factor approach [3,4].

The HSG-models have been constructed as integrable perturbations of WZNW-models. The related scattering matrices belong to a general class [5,6], which describe the scattering of particles labeled by two quantum numbers, where each of them may be associated to a simple Lie algebra. Characteristic features of these S-matrices are the breaking of the parity invariance of some amplitudes and in addition the presence of a resonance parameter which enables the formation of unstable bound states. In [1] we recovered the expected Virasoro coset central charge and found that when the resonance parameter tends to infinity the system decouples into several copies of minimal affine Toda field theories. Since the ultraviolet central charge is also accessible by the c-theorem, the findings in [1] may be checked for consistency.

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Our manuscript is organised as follows: In Section 2 we recall the general properties of form factors. In Section 3 we specialise the equations to the case of the $SU(3)_2$-HSG model and provide the general solutions related to the energy momentum tensor and the analogue of the order and disorder operators. Our conclusions and a further outlook are presented in Section 4.

2. Generalities on form factors

In order to fix our conventions and to set up the general framework we commence by recalling briefly some general properties of form factors. For a proper justification of them in terms of general principles of quantum field theory and analytic properties in the complex plane we refer the reader to [3,4,7,8].

Form factors are tensor valued functions, representing matrix elements of some local operator $\mathscr{O}(x)$ at the origin between a multiparticle in-state and the vacuum, which we denote by

$$F_n^{\mathscr{O}|\mu_1\dots\mu_n}(\theta_1,\dots,\theta_n) := \left\langle 0|\mathscr{O}(0)|V_{\mu_1}(\theta_1)V_{\mu_2}(\theta_2)\dots V_{\mu_n}(\theta_n)\right\rangle_{\text{in}} . \tag{1}$$

Here the $V_\mu(\theta)$ are some vertex operators representing a particle of species μ depending on the rapidity θ satisfying the so-called Zamolodchikov algebra.

As a consequence of CPT-invariance or the braiding of two operators $V_\mu(\theta)$ one obtains

$$F_n^{\mathscr{O}|\dots\mu_i\mu_{i+1}\dots}(\dots,\theta_i,\theta_{i+1},\dots) = F_n^{\mathscr{O}|\dots\mu_{i+1}\mu_i\dots}(\dots,\theta_{i+1},\theta_i,\dots)\,S_{\mu_i\mu_{i+1}}(\theta_{i,i+1}) . \tag{2}$$

As usual we abbreviate $\theta_{ij} = \theta_i - \theta_j$. The analytic continuation in the complex θ-plane at the cuts when $\theta = 2\pi i$ together with crossing leads to

$$F_n^{\mathscr{O}|\mu_1\dots\mu_n}(\theta_1 + 2\pi i,\dots,\theta_n) = F_n^{\mathscr{O}|\mu_2\dots\mu_n\mu_1}(\theta_2,\dots,\theta_n,\theta_1) . \tag{3}$$

Since we are describing relativistically invariant theories we expect for an operator $\mathscr{O}$ with spin s

$$F_n^{\mathscr{O}|\mu_1\dots\mu_n}(\theta_1 + \lambda,\dots,\theta_n + \lambda) = \mathrm{e}^{\lambda} F_n^{\mathscr{O}|\mu_1\dots\mu_n}(\theta_1,\dots,\theta_n) , \tag{4}$$

with λ being an arbitrary real number. For a form factor whose first two particles are conjugate to each other we have a kinematical pole at $i\pi$, which leads to a recursive equation relating the $(n-2)$- and the n-particle form factor

$$\operatorname*{Res}_{\bar\theta_0\to\theta_0} F_{n+2}^{\mathscr{O}|\bar\mu\mu\mu_1\dots\mu_n}(\bar\theta_0 + i\pi,\theta_0,\theta_1\dots,\theta_n) = i\left(1-\omega\prod_{l=1}^{n} S_{\mu\mu_l}(\theta_{0l})\right) F_n^{\mathscr{O}|\mu_1\dots\mu_n}(\theta_1,\dots,\theta_n), \tag{5}$$

with ω being the factor of local commutativity and $\bar\mu$ the anti-particle of μ. We restrict our initial considerations to a model in which stable bound states may not be formed and therefore we do not need to report the so-called bound state residue equation.

To be able to associate a solution of the Eqs. (2)–(5) to a particular operator, the following upper bound on the asymptotic behaviour [9]

$$\left[F_n^{\mathscr{O}|\mu_1\dots\mu_n}(\theta_1,\dots,\theta_n)\right]_i \le \Delta \tag{6}$$

turns out to be very useful. Here Δ denotes the conformal dimension of the operator $\mathscr{O}$ in the conformal limit. For convenience we introduced the short hand notation $\lim_{\theta_i\to\infty} f(\theta_1,\dots,\theta_n) = \text{const.}\cdot\exp([f(\theta_1,\dots,\theta_n)]_i\,\theta_i)$.

Ultimately form factors serve to compute correlation functions, but they may also be exploited to extract various other properties as for instance the difference between the ultraviolet and infrared Virasoro central charges, as stated in the so-called c-theorem [10]

$$\Delta c = \sum_{n=1}^{\infty} \sum_{\mu_1 \ldots \mu_n} \frac{9}{n!(2\pi)^n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{d\theta_1 \ldots d\theta_n}{\left(\sum_{i=1}^{n} m_{\mu_i} \cosh\theta_i \right)^4} \left| F_n^{\mathscr{O}|\mu_1 \ldots \mu_n}(\theta_1, \ldots, \theta_n) \right|^2 . \tag{7}$$

It is essentially the property (7) which we wish to employ for our purposes and check for consistency of the physical picture which emerged in [1].

3. The SU(3)$_2$-HSG model

For finite resonance parameter the SU(3)$_2$-HSG model describes the WZNW-coset model with central charge $c = \frac{6}{5}$ perturbed by an operator with conformal dimension $\Delta = \frac{3}{5}$. The model contains only two self-conjugate solitons which are conveniently denoted by "$+$" and "$-$", since that will allow for compact notations. The S-matrix elements read [5]

$$S_{\pm\pm} = -1 \quad \text{and} \quad S_{\pm\mp}(\theta) = \pm \tanh\frac{1}{2}\left(\theta \pm \sigma - i\frac{\pi}{2} \right). \tag{8}$$

This means the scattering of particles of the same type is simply described by the S-matrix of the thermal perturbation of the Ising model. Also the remaining amplitudes do not possess poles inside the physical sheet, such that the formation of stable particles via fusing is not possible. For vanishing resonance parameter σ the amplitudes $S_{\pm\mp}$ coincides formally with the ones which describe the massless flow between the tricritical Ising and the critical Ising model as analysed in [11]. However, there is an important conceptual difference since we view the expressions (8) as describing the scattering of massive particles. This has important consequences on the construction of the form factors and in fact the solution we compute below will be different from the one proposed in [11]. In the HSG setting the massless flow was recovered in the context of the thermodynamic Bethe ansatz [1] only as a subsystem in terms of specially introduced variables combining the inverse temperature and the resonance parameter. When the resonance parameter tends to infinity the amplitudes $S_{\pm\mp}$ become one, describing non-interacting scattering, such that the "$+$"-system and the "$-$"-system decouple.

Attempting now to solve the equations presented in Section 2, we proceed as usual in this context and we make a factorization ansatz which already extracts explicitly some of the singularity structure we expect to find. For the case at hand we have to have a kinematical pole at $i\pi$ when two particles are conjugate to each other

$$F_n^{\mathscr{O}|\overbrace{\mu_1 \ldots \mu_l}^{l\times\pm}\overbrace{\mu_{l+1} \ldots \mu_n}^{m\times\mp}}(\theta_1 \ldots \theta_n) = H_n^{\mathscr{O}|\mu_1 \ldots \mu_n} Q_n^{\mathscr{O}|\mu_1 \ldots \mu_n}(x_1 \ldots x_n) \prod_{i<j} \frac{F_{\min}^{\mu_i \mu_j}(\theta_{ij})}{\left(x_i^{\mu_i} + x_j^{\mu_j} \right)^{\delta_{\mu_i \mu_j}}} . \tag{9}$$

We introduced the variable $x_i = \exp\theta_i$. The $H_n^{\mathscr{O}|\mu_1 \ldots \mu_n}$ are normalization constants. As common we suppose that the so-called minimal form factor satisfies

$$F_{\min}^{ij}(\theta) = F_{\min}^{ji}(-\theta) S_{ij}(\theta) = F_{\min}^{ji}(2\pi i - \theta) \tag{10}$$

and has neither zeros nor poles in the physical sheet. Then, if we further assume that $Q_n^{\mathscr{O}|\mu_1 \ldots \mu_n}(\theta_1, \ldots, \theta_n)$ is separately symmetric in the first l and the last m rapidities and in addition $2\pi i$-periodic function in all rapidities, the ansatz (9) solves Watson's Eqs. (2) and (3) by construction. In particular we have

$$Q_n^{\mathscr{O}|\overbrace{\mu_1 \ldots \mu_l}^{l\times+}\overbrace{\mu_{l+1} \ldots \mu_n}^{m\times-}}(x_1, \ldots, x_n) = Q_n^{\mathscr{O}|\overbrace{\mu_{l+1} \ldots \mu_n}^{m\times-}\overbrace{\mu_1 \ldots \mu_l}^{l\times+}}(x_{l+1}, \ldots x_n, x_1, \ldots x_l), \tag{11}$$

such that when we have constructed a solution for one particular ordering of the μ's, e.g. the upper sign in (9), we can obtain the solution for a permuted ordering by the monodromy properties. Especially the reversed order we obtain by applying Eq. (11). Despite the fact that we do not gain anything new, it is still instructive to verify (5) as a consistency check also for the different ordering. The monodromy properties allow some simplification in the notation and from now on we restrict our attention w.l.g. to the upper sign in (9). In addition we deduce from Eq. (4) that for a spinless operator $\mathscr{O}$ the total degree of $Q_n^{\mathscr{O}}$ has to be $l(l-1)/2 - m(m-1)/2$.

A solution for the minimal form factors, i.e. of Eqs. (10), is found easily

$$F_{\min}^{\pm\pm}(\theta) = -i\sinh\frac{\theta}{2}, \tag{12}$$

$$F_{\min}^{\pm\mp}(\theta) = \mathscr{N}^{\pm}(\theta)\prod_{k=1}^{\infty}\frac{\Gamma\left(k+\frac{1}{4}\right)^2\Gamma\left(k+\frac{1}{4}+\frac{i}{2\pi}(\theta\pm\sigma)\right)\Gamma\left(k-\frac{3}{4}-\frac{i}{2\pi}(\theta\pm\sigma)\right)}{\Gamma\left(k-\frac{1}{4}\right)^2\Gamma\left(k-\frac{1}{4}-\frac{i}{2\pi}(\theta\pm\sigma)\right)\Gamma\left(k+\frac{3}{4}+\frac{i}{2\pi}(\theta\pm\sigma)\right)}, \tag{13}$$

$$= \mathscr{N}^{\pm}(\theta)\exp\left(-\int_0^\infty\frac{dt}{t}\,\frac{\sin^2\left((i\pi-\theta\mp\sigma)\frac{t}{2\pi}\right)}{\sinh t\cosh\frac{t}{2}}\right). \tag{14}$$

Here $F_{\min}^{\pm\pm}(\theta)$ is the well-known minimal form factor of the thermally perturbed Ising model [12,13] and for the upper choice of the signs, Eq. (14) coincides for vanishing σ up to normalization with the expression found in [11]. We introduced the normalization function $\mathscr{N}^{\pm}(\theta) = 2^{\frac{1}{4}}\exp\left(\frac{i\pi(1\mp1)\pm\theta}{4} - \frac{G}{\pi}\right)$ with $G = 0.91597$ being the Catalan constant. The minimal form factors possess various properties which we would like to employ in the course of our argumentation. They obey the functional identities

$$F_{\min}^{\pm\pm}(\theta+i\pi)F_{\min}^{\pm\pm}(\theta) = -\frac{i}{2}\sinh\theta, \tag{15}$$

$$F_{\min}^{\pm\mp}(\theta+i\pi)F_{\min}^{\pm\mp}(\theta) = \frac{i^{\frac{2\mp1}{2}}e^{\pm\frac{\theta}{2}}}{\cosh\frac{1}{2}\left(\theta\pm\sigma-\frac{i\pi}{2}\right)}. \tag{16}$$

We will also exploit the asymptotic behaviour

$$\lim_{\sigma\to\infty}F_{\min}^{\pm\mp}(\pm\theta)\sim e^{-\frac{\sigma}{4}},\quad \left[F_{\min}^{\pm\pm}(\theta_{ij})\right]_i = \frac{1}{2},\quad \left[F_{\min}^{\pm\mp}(\theta_{ij})\right]_i = \begin{cases}0\\ -\frac{1}{2}\end{cases}. \tag{17}$$

Together with the factorization ansatz (9) this leads us immediately to the relations

$$\left[F_n^{\mathscr{O}|l,m}\right]_i = \left[Q_n^{\mathscr{O}|l,m}\right]_i + \frac{1-l}{2}\quad \text{for}\quad 1\le i\le l \tag{18}$$

$$\left[F_n^{\mathscr{O}|l,m}\right]_i = \left[Q_n^{\mathscr{O}|l,m}\right]_i + \frac{m-l-1}{2}\quad \text{for}\quad l< i\le n, \tag{19}$$

which are useful in the identification process of a particular solution with a specific operator. Since we may restrict our attention to one particular ordering only, we abbreviate the r.h.s. of (9) from now on as $F_n^{\mathscr{O}|l,m}$ and similar for the Q's.

Substituting the ansatz (9) into the kinematic residue equation (5) reduces, with the help of (15) and (16), the whole problem of determining the form factors to the following recursive equations:

$$Q_{n+2}^{\mathcal{O}|l+2,m}(-x,x,\dots,x_n) = D_n^{l,m}(x_1,\dots,x_n)\,Q_n^{\mathcal{O}|l,m}(x_1,\dots,x_n)\,, \tag{20}$$

$$D_n^{l,m}(x,x_1,\dots,x_n) = \frac{1}{2}(-ix)^{l+1}\sigma_l^{+}\sum_{k=0}^{m}(-ie^{\sigma}x)^{-k}\left(1-\omega(-1)^{l+k}\right)\sigma_k^{-} \tag{21}$$

Here we introduced yet another short hand notation, namely for elementary symmetric polynomials $\sigma_k(x_1,\dots,x_l)\equiv\sigma_k^{+}$ and $\sigma_k(x_{l+1},\dots,x_n)\equiv\sigma_k^{-1}$. Below we shall also employ σ_k when the polynomials depend on all n variables, $\bar{\sigma}_k$ when they depend on the n inverse variables, i.e. $x_i^{-1}$and $\hat{\sigma}_k$ when they depend on the n variables $x_i e^{-\sigma}$.

The recursive equations for the constants turn out to be

$$H_{n+2}^{\mathcal{O}|l+2,m} = i^m 2^{2l-m+1}e^{\sigma m/2}H_n^{\mathcal{O}|l,m}\,. \tag{22}$$

Fixing one of the lowest constants, the solutions to these equations read

$$H^{\mathcal{O}|2s+t,m} = i^{sm}2^{s(2s-m-1+2t)}e^{sm\sigma/2}H^{\mathcal{O}|t,m}\,,\quad t=0,1\,. \tag{23}$$

For specific operators we will provide below the explicit expressions for the $H^{\mathcal{O}|l,m}$. Notice that there is a certain ambiguity contained in the Eqs. (22), i.e. we can multiply $H_n^{\mathcal{O}|l,m}$ by i^{2l}, i^{2l^2} or $(-1)^l$ and produce a new solution. However, since in practical applications we are usually dealing with the absolute values of the form factors, these ambiguities will turn out to be irrelevant.

3.1. Solutions

Whenever we consider $F_n^{\mathcal{O}|l,m}$ with l even for vanishing resonance parameter σ, we can use the kinematic residue equation (5) $l/2$-times and finally construct $F_n^{\mathcal{O}|0,m}$, which should correspond to a form factor of the thermally perturbed Ising model. In other words in that case we can always use the well-known solutions $Q_n^{\mathcal{O}|0,m}$ as the initial condition for the recursive problem (20).

3.1.1. The energy momentum tensor Θ

The only non-vanishing form factor of the energy momentum tensor in the thermally perturbed Ising model is well know to be

$$F_2^{\Theta}(\theta) = -2\pi i m^2\sinh(\theta/2)\,. \tag{24}$$

From this equation we deduce immediately that $[F_n^{\Theta|l,2}]_i = \frac{1}{2}$, which serves on the other hand to fix $[Q_n^{\Theta|l,2}]_i$ with the help of (18) and (19). Recalling that the energy momentum tensor is proportional to the perturbing field [15] and the fact that the conformal dimension of the latter is $\Delta=\frac{3}{5}$ for the $SU(2)_3$-HSG model, the value $[F_n^{\Theta|l,2}]_i=\frac{1}{2}$ is compatible with the bound (6). As a further consequence of (24), we deduce

$$H^{\Theta|0,2} = 2\pi m_-^2 \tag{25}$$

as the initial value for the computation of all higher constants in (23). The distinction between m_- and m_+ indicates that in principle the mass scales could be very different as discussed in [1]. Notice that $H^{\Theta|0,0}$ is

[1] The elementary symmetric polynomials are generated by

$$\prod_{k=1}^{n}(x+x_k)=\sum_{k=0}^{n}x^{n-k}\sigma_k(x_1,\dots,x_n),\quad\text{i.e.}\quad \sigma_k(x_1,\dots,x_n)=\frac{1}{2\pi i}\oint\frac{dz}{z^{n-k+1}}\prod_{k=1}^{n}(z+x_k)$$

(For more properties see e.g. [14].)

reached only formally, since the kinematic residue equation does not connect to the vacuum expectation value. The initial values for the recursive Eqs. (20) are

$$Q_2^{\Theta|0,2} = x_1^{-1} + x_2^{-1} \quad \text{and} \quad Q_{2t}^{\Theta|0,2t} = 0 \quad \text{for } t \geq 2 . \tag{26}$$

Taking now $\omega = 1$, the solutions to (20), with the same asymptotic behaviour as the energy momentum tensor in the thermally perturbed Ising model, are computed to

$$Q_{2s+2t}^{\Theta|2s,2t} = (-1)^{(s+1)t} \mathrm{e}^{-t\sigma} \sigma_1 \bar{\sigma}_1 (\sigma_{2s}^{+})^{s-t} (\sigma_{2t}^{-})^{1-t} \det \mathscr{A}^{\Theta} \quad \text{for } t \geq 1, s \geq 1, \tag{27}$$

where $\mathscr{A}^{\Theta}$ is a $(t+s-2) \times (t+s-2)$-matrix whose entries are given by

$$\mathscr{A}_{ij}^{\Theta} = \begin{cases} \sigma_{2(j-i)+1}^{+} & \text{for} \quad 1 \leq i < t \\ (-1)^{(j-i+t)} \hat{\sigma}_{2(j-i+t)-1}^{-} & \text{for} \quad t \leq i \leq s+t-2 \end{cases} . \tag{28}$$

Explicitly we have

$$\mathscr{A}^{\Theta} = \begin{pmatrix} \sigma_1^{+} & \sigma_3^{+} & \sigma_5^{+} & \sigma_7^{+} & \cdots & 0 \\ 0 & \sigma_1^{+} & \sigma_3^{+} & \sigma_5^{+} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \sigma_{2s-1}^{+} \\ -\hat{\sigma}_1^{-} & \hat{\sigma}_3^{-} & -\hat{\sigma}_5^{-} & \hat{\sigma}_7^{-} & \cdots & 0 \\ 0 & -\hat{\sigma}_1^{-} & \hat{\sigma}_3^{-} & -\hat{\sigma}_5^{-} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & (-1)^t \hat{\sigma}_{2t-1}^{-} \end{pmatrix} . \tag{29}$$

One may easily verify case-by-case that (27) is a solution of (22) to relatively high orders in s and t. A general proof of this result, which we present elsewhere [16], can be obtained by exploiting the fact that the determinant of $\mathscr{A}$ may also be represented in terms contour integrals

$$\begin{aligned} \det \mathscr{A}^{\Theta} &= (-1)^{(s+1)t} \oint du_1 \ldots \oint du_{t-1} \oint dv_1 \ldots \oint dv_{s-1} \prod_{j=1}^{t-1} u_j^{2-2s-2j} \prod_{i=1}^{2s} (u_j + x_i) \prod_{j=1}^{s-1} v_j^{2-2t-2j} \\ &\quad \times \prod_{i=1+2s}^{2s+2t} (v_j + \hat{x}_i) \prod_{1 \leq i < j \leq t-1} (u_j^2 - u_i^2) \prod_{1 \leq i < j \leq s-1} (v_j^2 - v_i^2) \prod_{j=1}^{s-1} \prod_{i=1}^{t-1} (u_i^2 + v_j^2) . \end{aligned} \tag{30}$$

In order to establish the equivalence between (29) and (30) we simply use the integral representation for the symmetric polynomals as stated in the footnote. The integrals in (30) are understood as $\oint dz \equiv (2\pi i)^{-1} \oint_{|z|=\varrho} dz$ with ϱ being an arbitrary positive real number.

Assembling now all the quantities we obtain for instance

$$F_4^{\Theta|++--}(\theta_1,\theta_2,\theta_3,\theta_4) = \frac{-\pi m_-^2 \mathrm{e}^{(\theta_{31}+\theta_{42})/2} \left(2 + \sum_{i<j} \cosh(\theta_{ij})\right)}{2\cosh(\theta_{12}/2)\cosh(\theta_{34}/2)} \prod_{i<j} F_{\min}^{\mu_i \mu_j}(\theta_{ij}) . \tag{31}$$

Having computed all form factors for the energy momentum tensor we are in the position to apply the c-theorem, i.e. we can in principle evaluate (7). For finite values of σ we obtain

$$\Delta c^{(2)} = 1, \quad \Delta c^{(4)} = 1.197 \ldots, \quad \Delta c^{(6)} = 1.199 \ldots, \quad \text{for } \sigma < \infty , \tag{32}$$

where in the notation $\Delta c^{(n)}$, the superscript n indicates the upper limit in (7). Thus, the expected value of $c = \frac{6}{5}$ is well reproduced. Apart from $\Delta c^{(2)}$, in which case the calculation can be performed analytically, the integrals in (7) are computed directly by a brute force Monte Carlo integration.

When the resonance parameter tends to infinity the system decouples and we are left with two non-interacting free fermions, such that the only contribution in the sum (7) is twice the free fermion two-particle contribution, such that

$$\lim_{\sigma\to\infty} \Delta c = 1\,. \tag{33}$$

In order to see this we collect the leading order behaviours form our general solution

$$\lim_{\sigma\to\infty} H_{2s+2t}^{\Theta|2s,2t} \sim e^{st\sigma}, \quad \lim_{\sigma\to\infty} Q_{2s+2t}^{\Theta|2s,2t} \sim e^{-(t+s-1)\sigma}, \quad \lim_{\sigma\to\infty} \prod_{i<j} F_{\min}^{\mu_i\mu_j}(\theta_{ij}) \sim e^{-st\sigma}, \tag{34}$$

which means

$$\lim_{\sigma\to\infty} F_{2s+2t}^{\Theta|2s,2t} \sim e^{-(t+s-1)\sigma}\,. \tag{35}$$

Hence the only non-vanishing form factors in this limit are $F_2^{\Theta|0,2}$ and $F_2^{\Theta|2,0}$, which establishes (33).

3.1.2. The order operator Σ

For the other sectors we may proceed similarly, i.e. viewing always the thermally perturbed Ising model as a benchmark. Taking now $\omega = 1$, we recall the solution for the order operator

$$F_{2s+1}^{\Sigma}(\theta_1,\ldots,\theta_{2s+1}) = i^s F_1^{\Sigma} \prod_{i<j} \tanh\frac{\theta_{ij}}{2} = i^s (2i)^{2s^2+s} F_1^{\Sigma}(\sigma_{2s+1})^s \prod_{i<j} \frac{F_{\min}^{\pm\pm}(\theta_{ij})}{x_i + x_j}\,. \tag{36}$$

With this information we may fix the initial values of the recursive Eqs. (20) and (22) at once to

$$Q_{2t+1}^{\Sigma|0,2t+1} = (\sigma_{2t+1})^{-t} = (\bar{\sigma}_{2t+1})^{t} \quad \text{and} \quad H^{\Sigma|0,1} = F_1^{\Sigma}\,. \tag{37}$$

Furthermore, we deduce from Eq. (36) that $[F_n^{\Sigma|2s,2t+1}]_i = 0$. Respecting these constraints we find as explicit solutions

$$Q_{2s+2t+1}^{\Sigma|2s,2t+1} = (-1)^{(s+1)t} (\sigma_1)^{\frac{1}{2}} (\sigma_{2s}^{+})^{s-t-1} (\sigma_1^{-})^{-\frac{1}{2}} (\sigma_{2t+1}^{-})^{-t} \det \mathscr{A}^{\Sigma}\,, \tag{38}$$

where $\mathscr{A}^{\Sigma}$ is a $(t+s)\times(t+s)$-matrix whose entries are given by

$$\mathscr{A}_{ij}^{\Sigma} = \begin{cases} \sigma_{2(j-i)}^{+} & \text{for} \quad 1 \le i \le t \\ (-1)^{(j-i+t+1)} \hat{\sigma}_{2(j-i+t)+1}^{-} & \text{for} \quad t < i \le s+t \end{cases}. \tag{39}$$

Explicitly this reads

$$\mathscr{A}^{\Sigma} = \begin{pmatrix} 1 & \sigma_2^{+} & \sigma_4^{+} & \sigma_6^{+} & \cdots & 0 \\ 0 & 1 & \sigma_2^{+} & \sigma_4^{+} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \sigma_{2s}^{+} \\ -\hat{\sigma}_1^{-} & \hat{\sigma}_3^{-} & -\hat{\sigma}_5^{-} & \hat{\sigma}_7^{-} & \cdots & 0 \\ 0 & -\hat{\sigma}_1^{-} & \hat{\sigma}_3^{-} & -\hat{\sigma}_5^{-} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & (-1)^{(t+1)} \hat{\sigma}_{2t+1}^{-} \end{pmatrix}. \tag{40}$$

Once again the determinant of $\mathscr{A}$ admits an integral representation

$$\det \mathscr{A}^{\Sigma} = (-1)^{s(t-1)} \oint du_1 \ldots \oint du_t \oint dv_1 \ldots \oint dv_s \prod_{i=1}^{2s} u_j^{1-2s-2j} \prod_{j=1}^{t} (u_j + x_i) \prod_{i=1+2s}^{2s+2t+1} v_j^{2-2t-2j}$$
$$\times \prod_{j=1}^{s} (v_j + \hat{x}_i) \prod_{1 \le i < j \le t} (u_j^2 - u_i^2) \prod_{1 \le i < j \le s} (v_j^2 - v_i^2) \prod_{j=1}^{s} \prod_{i=1}^{t} (u_i^2 + v_j^2) \tag{41}$$

which may be used for a general proof [16].

When the resonance parameter tends to infinity we obtain the following asymptotic behaviour

$$\lim_{\sigma \to \infty} Q_{2s+2t+1}^{\mu|2s,2t+1} \sim e^{-s\sigma} \tag{42}$$

$$\lim_{\sigma \to \infty} H_{2s+2t+1}^{\mu|2s,2t+1} \prod_{i<j} F_{\min}^{\mu_i \mu_j}(\theta_{ij}) = \text{const} \prod_{1 \le i < j \le 2s} F_{\min}^{++}(\theta_{ij}) \prod_{2s < i < j \le 2s+2t+1} F_{\min}^{--}(\theta_{ij}) . \tag{43}$$

This means unless $s = 0$, that is a reduction to the thermally perturbed Ising model, the form factors will vanish in this limit.

3.1.3. The disorder operator μ

For the disorder operator we have $\omega = -1$ and the solution acquires the same form as in the previous case

$$F_{2s}^{\mu}(\theta_1, \ldots, \theta_{2s}) = i^s F_0^{\mu} \prod_{i<j} \tanh \frac{\theta_{ij}}{2} . \tag{44}$$

Similar as for the order variable we can fix the initial values of the recursive Eqs. (20) and (22) to

$$Q_{2t}^{\mu|0,2t} = (\sigma_{2t})^{1/2-t} = (\bar{\sigma}_{2t})^{t-1/2} \quad \text{and} \quad H^{\mu|0,0} = F_0^{\mu} . \tag{45}$$

Furthermore, we deduce $[F_n^{\mu|2s,2t}]_i = 0$. Respecting these constraints we find as a general solution

$$Q_{2s+2t}^{\mu|2s,2t} = (-1)^{st} (\sigma_{2s+2t})^{\frac{3}{2}-t} (\sigma_{2s}^{+})^{s-2} (\sigma_{2t}^{-})^{-1} \det \mathscr{A}^{\mu} , \tag{46}$$

where $\mathscr{A}^{\mu}$ is a $(t+s) \times (t+s)$-matrix whose entries are given by

$$\mathscr{A}_{ij}^{\mu} = \begin{cases} \sigma_{2(j-i)}^{+} & \text{for} \quad 1 \le i \le t \\ (-1)^{(j-i+t)} \hat{\sigma}_{2(j-i+t)}^{-} & \text{for} \quad t < i \le s+t \end{cases} . \tag{47}$$

Explicitly we have

$$\mathscr{A}^{\mu} = \begin{pmatrix} 1 & \sigma_2^{+} & \sigma_4^{+} & \sigma_6^{+} & \cdots & 0 \\ 0 & 1 & \sigma_2^{+} & \sigma_4^{+} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \sigma_{2s}^{+} \\ 1 & -\hat{\sigma}_2^{-} & \hat{\sigma}_4^{-} & -\hat{\sigma}_6^{-} & \cdots & 0 \\ 0 & 1 & -\hat{\sigma}_2^{-} & \hat{\sigma}_4^{-} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & (-1)^{t} \hat{\sigma}_{2t}^{-} \end{pmatrix} . \tag{48}$$

Similarly as in the previous sections we can write the determinant of $\mathscr{A}$ alternatively in form of an integral representation

$$\det \mathscr{A}^{\mu} = (-1)^{s(t-1)} \oint du_1 \dots \oint du_t \oint dv_1 \dots \oint dv_s \prod_{i=1}^{2s} u_j^{1-2s-2j} \prod_{j=1}^{t} (u_j + x_i) \prod_{i=1+2s}^{2s+2t} v_j^{1-2t-2j}$$
$$\times \prod_{j=1}^{s} (v_j + \hat{x}_i) \prod_{1 \le i < j \le t} (u_j^2 - u_i^2) \prod_{1 \le i < j \le s} (v_j^2 - v_i^2) \prod_{j=1}^{s} \prod_{i=1}^{t} (u_i^2 + v_j^2). \tag{49}$$

When the resonance parameter tends to infinity we observe the following asymptotic behaviour:

$$\lim_{\sigma \to \infty} Q_{2s+2t}^{\mu|2s,2t} = (-1)^{st} Q_{2s}^{\mu|2s,0} Q_{2t}^{\mu|0,2t} \tag{50}$$

$$\lim_{\sigma \to \infty} H_{2s+2t}^{\mu|2s,2t} \prod_{i<j} F_{\min}^{\mu_i \mu_j}(\theta_{ij}) = \text{const.} \prod_{1 \le i < j \le 2s} F_{\min}^{++}(\theta_{ij}) \prod_{2s < i < j \le 2t+2s} F_{\min}^{--}(\theta_{ij}) \tag{51}$$

such that

$$\lim_{\sigma \to \infty} F_{2s+2t}^{\mu|2s,2t} \sim F_{2t}^{\mu|0,2t} F_{2s}^{\mu|2s,0} . \tag{52}$$

This means also in this sector we observe the decoupling of the theory into two free fermions.

4. Conclusions

The application of the c-theorem confirms very well the physical picture we found in [1] from the thermodynamic Bethe ansatz. For finite resonance parameter we recover the expected Virasoro central charge of $c = \frac{6}{5}$ and for $\sigma \to \infty$ the theory decouples in all sectors into two non-interacting free fermions. Besides the construction of all n-particle form factors related to the trace of energy momentum, we computed in addition the complete solutions for the order and disorder operator in form of determinants whose entries are symmetric polynomials. Such determinant formulae have occurred before in various places in the literature, e.g. [7,17]. Representing the solutions for form factors in this form has turned out to be useful in the construction of correlation functions [18] and might eventually lead to a reformulation of the whole problem in terms of differential equations analogous to the situation in conformal field theory [19]. Apart from higher spin solutions which may always be constructed by including the polynomials as suggested in [20], we did not find any additional solutions related to other sectors. We expect that a careful analysis of the cluster decomposition property will lead to more conclusive statements concerning the question whether such solutions exist at all. From a mathematical point of view it is also desirable to present a rigorous proof of the determinant formulae [16].

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References

[1] O.A. Castro-Alvaredo, A. Fring, C. Korff, J.L. Miramontes, Thermodynamic Bethe Ansatz of the Homogeneous Sine-Gordon Models, Nucl. Phys. B in press, hep-th/9912196.
[2] C.R. Fernández-Pousa, M.V. Gallas, T.J. Hollowood, J.L. Miramontes, Nucl. Phys. B 484 (1997) 609; Q.-H. Park, Phys. Lett. B 328 (1994) 329; T.J. Hollowood, J.L. Miramontes, Q.-H. Park, Nucl. Phys. B 445 (1995) 451.
[3] P. Weisz, Phys. Lett. B 67 (1977) 179; M. Karowski, P. Weisz, Nucl. Phys. B 139 (1978) 445.
[4] F.A. Smirnov, Form factors in Completely Integrable Models of Quantum Field Theory, Adv. Series in Math. Phys. 14, World Scientific, Singapore, 1992.
[5] J.L. Miramontes, C.R. Fernández-Pousa, Phys. Lett. B 472 (2000) 392.
[6] A. Fring, C. Korff, Phys. Lett. B 477 (2000) 380.
[7] Al.B. Zamolodchikov, Nucl. Phys. B 348 (1991) 619.
[8] H. Babujian, A. Fring, M. Karowski, A. Zapletal, Nucl. Phys. B [FS] 538 (1999) 535.
[9] G. Delfino, G. Mussardo, Nucl. Phys. B 455 (1995) 724.
[10] A.B. Zamolodchikov, JETP Lett. 43 (1986) 730.
[11] G. Delfino, G. Mussardo, P. Simonetti, Phys. Rev. D 51 (1995) R6620.
[12] B. Berg, M. Karowski, P. Weisz, Phys. Rev. D 19 (1979) 2477.
[13] V.P. Yurov, Al.B. Zamolodchikov, Int. J. Mod. Phys. A 6 (1991) 3419.
[14] I.G. MacDonald, Symmetric Functions and Hall Polynomials, Clarendon Press, Oxford, 1979.
[15] J.L. Cardy, Phys. Rev. Lett. 60 (1988) 2709.
[16] O.A. Castro-Alvaredo, A. Fring, C. Korff, J.L. Miramontes, in preparation.
[17] A. Fring, G. Mussardo, P. Simonetti, Nucl. Phys. B 393 (1993) 413; Phys. Lett. B 307 (1993) 83; A. Koubek, G. Mussardo, Phys. Lett. B 311 (1993) 193.
[18] V.E. Korepin, N.A. Slavnov, J. Phys. A 31 (1998) 9283; V.E. Korepin, T. Oota, J. Phys. A 31 (1998) L371; T. Oota, J. Phys. A 31 (1998) 7611.
[19] A.A. Belavin, A.M. Polyakov, A.B. Zamolodchikov, Nucl. Phys. B 241 (1984) 333.
[20] J.L. Cardy, G. Mussardo, Nucl. Phys. B 340 (1990) 387.