

# AS1051: Mathematics

## 0. Introduction

The aim of this course is to review the basic mathematics which you have already learnt during A-level, and then develop it further. You should find it almost entirely familiar, with only the occasional topic of new material.

However, even for those parts which are familiar there will be aspects which will be treated differently at university level. Most importantly, you will be expected to *know* key facts and formulas — you should not expect to have a formula sheet at your disposal. Thus, for example, you will be expected to memorise all the standard integrals and derivatives, the trigonometric identities, etc.

There are several books recommended for this course:

**Core maths: for advanced level**

L. Bostock, S. Chandler (Nelson Thornes, 2000)

City library shelfmark: 510 BOS

**Further pure mathematics**

L. Bostock, S. Chandler, C. Rourke

(Cheltenham : Thornes, 1982)

City library shelfmark: 510 BOS

**Understanding pure mathematics**

A.J. Sadler, D.W.S. Thorning

(Oxford: Oxford University Press, 1987)

City library shelfmark: 510 SAD

**A course in pure mathematics**

M.M. Gow

(Hodder and Stoughton, 1960)

City library shelfmark: 510 GOW

All books are available in the City University Library.

You will also notice that the pace of university mathematics is much faster than at school. This will certainly be true of the revision of A-level material, but will also extend to the new material. In part this will be because there will be fewer worked examples; you will be expected to practise calculations by yourself. Also, if you do not keep up to date, the speed of the course will make it hard for you to catch up.

As in all courses it is important that you attempt the exercise sheets. These will not be marked, but without working through them you are very unlikely to perform well in the final exams. Tutors are very pleased when students ask questions about material they do not understand — you should make full use of them!

**Course material:**

The entire course material is available on the course web site:

<http://www.staff.city.ac.uk/~fring/ActMath/Web/index.html>

This includes the lecture notes, exercise sheets, course work sheets, past papers and some relevant links. The page is regularly updated.

There is a link to this page from your moodle account, but you still need that account to check your marks.

**Assessment:**

There will be one course work and one progress test both with a pass mark of 40% .

## 1. Arithmetic

In this chapter we will review the basic algebraic manipulations which should already be familiar. First we introduce the main classes of numbers.

### 1.1 Numbers

Most basic are the **natural numbers**,  $\mathbb{N}$ , which consist of the positive whole numbers 1, 2, 3, ... (Some textbooks include 0 as a natural number, often referred to as  $\mathbb{N}_0$ ) Note in passing that **positive** means  $> 0$ , and **negative** means  $< 0$ . To talk about numbers  $\geq 0$  we say **non-negative**.

The **integers**  $\mathbb{Z}$  consist of all whole numbers 0,  $\pm 1$ ,  $\pm 2$ , ...

An integer  $a$  is **divisible** by another (non-zero) integer  $b$  if there exists a third integer  $c$  such that  $a = bc$ . In this case we call  $b$  a **divisor** of  $a$ . An integer  $p$  is **prime** if  $p > 1$  and  $p$  has no positive divisors except 1 and  $p$ .

Although easy to define, integers are hard to completely understand. For example, we do not have a formula for determining quickly whether a given number is prime.

Primes are important because of

**Theorem 1.1.1: (The fundamental theorem of arithmetic)**

*Every positive integer has a unique prime factorisation.*

Note that this results says two things: there **is** a factorisation as a product of primes, and it is **unique**.

**Example 1.1.2:**  $2,522,520 = 2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 7 \times 7 \times 11 \times 13$ .

Given two non-zero integers  $m$  and  $n$  we define their **highest common factor**  $\text{hcf}(m, n)$  to be the largest divisor of  $m$  and  $n$ , and the **least common multiple**  $\text{lcm}(m, n)$  to be the smallest positive integer divisible by  $m$  and  $n$ .

**Example 1.1.3:** If  $m = 60 = 2 \times 2 \times 3 \times 5$  and  $n = 70 = 2 \times 5 \times 7$ , then  $\text{hcf}(m, n) = 10$  and  $\text{lcm}(m, n) = 420$ .

The **rational numbers**,  $\mathbb{Q}$ , consist of all numbers of the form  $r = \frac{p}{q}$  where  $p$  and  $q$  are integers with  $q \neq 0$ . Note that there are equivalent forms of a rational number:

$$\frac{p}{q} = \frac{s}{t} \quad \text{if and only if} \quad pt = qs.$$

**Example 1.1.4:**  $\frac{3}{5} = \frac{9}{15}$  as  $3 \times 15 = 5 \times 9$ .

We usually simplify fractions to the form  $r = \frac{p}{q}$  where  $\text{hcf}(p, q) = 1$ . (Integers with  $\text{hcf}(p, q) = 1$  are called **coprime**.)



## 1.4 Permutations and combinations

Suppose we have a collection of  $n$  distinct objects. We can ask how many ways we can choose  $r$  objects from them if

- we do care what order we choose them in;
- we do not care what order we choose them in.

The first case is called the number of **permutations** and the second the number of **combinations**.

**Example 1.4.1:** From 1, 2, 3 we have six permutations of two elements

1, 2    1, 3    2, 1    2, 3    3, 1    3, 2

and the following three combinations

1, 2    1, 3    2, 3.

In general the number of permutations of  $r$  objects from a set of  $n$  distinct objects is given by

$${}_nP_r = \frac{n!}{(n-r)!}$$

and the number of combinations is just  ${}_nP_r / r!$ , which equals

$${}_nC_r = \frac{n!}{r!(n-r)!}.$$

Note that this is the same as the coefficient in the binomial theorem.

lecture 3

## 1.5 Polynomials

A **polynomial of degree  $n$**  in  $x$  is a function  $p(x)$  of the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

where  $a_0, a_1, \dots, a_n$  are constants with  $a_n \neq 0$ .

We call degree 2 polynomials **quadratic**, degree 3 **cubic** etc.

Traditionally we write quadratics in the generic form

$$ax^2 + bx + c. \quad (2)$$

To **complete the square** we write a quadratic in the form

$$a((x+d)^2 + e). \quad (3)$$

for some constants  $a, d$ , and  $e$  related to  $b, c$ . In this case the roots of the polynomial (3) (if they exist) are given by

$$x = -d \pm \sqrt{-e}. \quad (4)$$

If  $a$  is positive (respectively negative) then the minimum (respectively maximum) occurs at  $x = -d$  equalling  $ae$ .

**Example 1.5.1:** We will complete the square for the following quadratic.

$$\begin{aligned} f(x) &= 3x^2 + 2x - 4 \\ &= 3\left(x^2 + \frac{2x}{3} - \frac{4}{3}\right) \\ &= 3\left(\left(x + \frac{1}{3}\right)^2 - \frac{13}{9}\right) \end{aligned}$$

has roots  $x = -\frac{1}{3} \pm \sqrt{\frac{13}{9}}$  and a minimum at  $x = -\frac{1}{3}$  of  $-\frac{13}{3}$ .

Comparing coefficients in (2) and (3) we find  $b = 2ad$  and  $c = ad^2 + ae$  and therefore we deduce with (4) the well known formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

for the roots of  $ax^2 + bx + c$ . So we have

- two distinct roots if  $b^2 - 4ac > 0$ ,
- one root if  $b^2 - 4ac = 0$ ,
- no roots if  $b^2 - 4ac < 0$ .

If we denote the roots by  $\alpha$  and  $\beta$  then we have

$$f(x) = a(x - \alpha)(x - \beta) = ax^2 + bx + c$$

and so

$$a(x^2 - (\alpha + \beta)x + \alpha\beta) = ax^2 + bx + c.$$

From this we deduce that

$$\alpha + \beta = -\frac{b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a}.$$

Similar formulae can be deduced for cubics, quartics, etc.

**Example 1.5.2:** If the roots of  $x^2 + 5x + 3 = 0$  are  $\alpha$  and  $\beta$ , find the quadratic equation with roots  $\alpha^3$  and  $\beta^3$ .

We have  $\alpha + \beta = -5$  and  $\alpha\beta = 3$ . So

$$\begin{aligned} \alpha^3 + \beta^3 &= (\alpha + \beta)^3 - 3\alpha^2\beta - 3\alpha\beta^2 \\ &= -125 - 3\alpha\beta(\alpha + \beta) \\ &= -125 - 9(-5) = -80. \end{aligned}$$

and  $\alpha^3\beta^3 = (\alpha\beta)^3 = 27$ . Thus the required equation is

$$x^2 + 80x + 27 = 0.$$

Returning to general polynomials, we can easily add and multiply them to form new polynomials. However,  $p(x)/q(x)$  is not in general a polynomial.

and therefore  $p(x)/q(x)$  equals

$$\frac{x^2 + 1}{x - 2} = x + 2 + \frac{5}{x - 2}.$$

More generally: Let  $p(x)$  be a polynomial of degree  $n$  and divide  $p(x)$  by  $x - a$ , where  $a$  is a constant:

$$\frac{p(x)}{x - a} = q(x) + \frac{r}{x - a}$$

where  $q(x)$  is a polynomial and  $r$  is a constant, i.e.

$$p(x) = (x - a)q(x) + r.$$

From this we deduce

**Fact:** Every polynomial can be factorised into linear and/or quadratic terms.

If  $p(x) = a_n x^n + \dots + a_0$  has  $n$  distinct roots  $x_1, \dots, x_n$ , then

$$p(x) = a_n(x - x_1)(x - x_2) \cdots (x - x_n).$$

#### lecture 4

#### The method of undetermined coefficients

If two polynomials are identical — i.e. are equal for every value of  $x$  — then the coefficients of like terms are equal.

**Example 1.5.6:** Find  $a, b, c, d$  such that

$$r^3 = ar(r - 1)(r - 2) + br(r - 1) + cr + d.$$

Expanding we see that

$$\begin{aligned} r^3 &= a(r^3 - 3r^2 + 2r) + b(r^2 - r) + cr + d \\ &= ar^3 + (b - 3a)r^2 + (2a - b + c)r + d. \end{aligned}$$

Therefore  $a = 1$ ,  $b - 3a = 0$ ,  $2a - b + c = 0$  and  $d = 0$ ; i.e.  $a = 1$ ,  $b = 3$ ,  $c = 1$ , and  $d = 0$ .

In fact we used this method already before.

**Example 1.5.3:** Let  $p(x) = x^2 + 1$  and  $q(x) = x - 2$ . Then

$$p(x) + q(x) = x^2 + x - 1$$

and

$$p(x)q(x) = (x^2 + 1)(x - 2) = x^3 - 2x^2 + x - 2.$$

For  $p(x)/q(x)$  we have

$$\begin{array}{r} x + 2 \\ x - 2 \overline{) x^2 + 0x + 1} \\ \underline{x^2 - 2x} \phantom{+ 1} \\ 2x + 1 \\ \underline{2x - 4} \\ 5 \end{array}$$

**Theorem 1.5.4:** If  $p(x)$  is a polynomial with  $p(a) = r$  then

$$p(x) = (x - a)q(x) + r$$

for some polynomial  $q(x)$ .

When  $r \neq 0$  this is called the **remainder theorem** and when  $r = 0$  it is called the **factor theorem**.

**Example 1.5.5:** Factorise

$$f(x) = x^3 - 7x^2 + 7x + 15.$$

We try some numbers:  $f(0) = 15$ ,  $f(1) = 16$ ,  $f(-1) = 0$ , and so  $x + 1$  is a factor.

$$\begin{aligned} f(x) &= (x + 1)(x^2 - 8x + 15) \\ &= (x + 1)(x - 3)(x - 5). \end{aligned}$$

Standard results (to be **memorised**):

$$\begin{aligned} x^2 - a^2 &= (x - a)(x + a) \\ x^3 - a^3 &= (x - a)(x^2 + ax + a^2) \\ x^n - a^n &= (x - a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \dots + a^{n-2}x + a^{n-1}) \end{aligned}$$

When  $n$  is odd we can get a formula for  $x^n + a^n$  from the last one by replacing  $a$  by  $(-a)$ . However, there is no simple formula for the case  $n$  even.

#### 1.6 Rational functions

A **rational function** is a function of the form  $\frac{p(x)}{q(x)}$  where  $p(x)$  and  $q(x)$  are polynomials with  $q(x)$  not identically zero. (That is, there is at least one value of  $x$  for which  $q(x) \neq 0$ .) For example

$$\frac{x^2 + 6x + 4}{x^2 - 5} \quad \text{and} \quad \frac{x + 7}{3x^7 - 2x + 1}.$$

We can add or subtract rational functions just like we do ordinary fractions. We can also simplify them in the same way (by removing common factors from the top and bottom).

A **proper** rational function is one where the degree of the numerator is less than the degree of the denominator. Otherwise we say the function is **improper**. For example, the first fraction above is improper, the second proper.

There is a second way to simplify a rational function which has a product of factors in the denominator, using partial fractions. To do this to a fraction  $p(x)/q(x)$  we use the following procedure:

**Step 1:** Simplify  $p(x)/q(x)$  to form a proper rational function.

**Step 2:** Factorise the denominator into linear and quadratic factors.

**Step 3:** If  $q$  has  $r$  factors, write the fraction as a sum of  $n$  terms using the following correspondence between factors of  $q$  and summands:

$$(x-a)^r \longleftrightarrow \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_r}{(x-a)^r}$$

$$(ax^2+bx+c)^r \longleftrightarrow \frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_rx+B_r}{(ax^2+bx+c)^r}$$

where  $A_i$  and  $B_i$  (with  $1 \leq i \leq r$ ) are constants.

The total number of constants equals the degree of the denominator. These constants can be determined by using the method of undetermined coefficients.

#### Example 1.6.1:

$$\frac{x+5}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}.$$

Therefore

$$x+5 = A(x+1) + B(x-3).$$

We could equate coefficients, instead we substitute values chosen in such a way that most terms disappear. Substituting  $x = -1$  and  $x = 3$  we obtain

$$4 = -4B \quad \text{and} \quad 8 = 4A$$

i.e.  $A = 2$  and  $B = -1$ .

#### Example 1.6.3:

$$\frac{5x-12}{(x+2)(x^2-2x+3)} = \frac{A}{x+2} + \frac{Bx+C}{x^2-2x+3}$$

as  $x^2 - 2x + 3$  cannot be factorised. Therefore

$$5x-12 = A(x^2-2x+3) + (Bx+C)(x+2).$$

Substituting  $x = -2$  we obtain  $A = -2$ . By comparing coefficients of the  $x^2$  terms and constant terms we obtain  $B = 2$  and  $C = -3$ .

#### Example 1.6.2:

$$\frac{2x^2+x-2}{x^3(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-1}.$$

Therefore

$$2x^2+x-2 = Ax^2(x-1) + Bx(x-1) + C(x-1) + Dx^3.$$

Substituting  $x = 0$  and  $x = 1$  we obtain

$$-2 = -C \quad \text{and} \quad 1 = D$$

Comparing coefficients of the  $x^3$  terms and the  $x^2$  terms we obtain

$$0 = A + D \quad \text{and} \quad 2 = -A + B$$

and hence  $A = -1$ ,  $B = 1$ ,  $C = 2$ ,  $D = 1$ .

#### Example 1.6.4:

$$\begin{aligned} \frac{3x^3-x^2+2}{x(x^2-1)} &= \frac{3(x^3-x)-x^2+3x+2}{x(x-1)(x+1)} \\ &= 3 - \frac{(x^2-3x-2)}{x(x-1)(x+1)}. \end{aligned}$$

Now

$$\frac{(x^2-3x-2)}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

and we can show that  $A = 2$ ,  $B = -2$  and  $C = 1$ . Therefore

$$\frac{3x^3-x^2+2}{x(x^2-1)} = 3 - \frac{2}{x} + \frac{2}{x-1} - \frac{1}{x+1}.$$