7. Calculus III: Limits

7.1 The limit of a function

We will give an informal definition of the limit of a function (a concept which we have already used in this course). For a rigorous treatment of this subject we would need a precise definition and lots of proofs, but we will make do with an informal survey of the main results and methods.

There will be a brief introduction to the formal definition of a limit at the end of the Chapter.

Informally, given a function f and a number c, the limit of f as x tends to c, written $\lim_{x \to c} f(x)$ is defined in the following way.

- $\lim_{x \to c} f(x) = L$
  if the values of $f(x)$ are always arbitrarily close to L provided that x is sufficiently close (but not equal) to c.
- $\lim_{x \to c} f(x) = \infty$
  if for every given real R number, $f(x)$ is always bigger than R, provided that x is sufficiently close to c.
- $\lim_{x \to c} f(x) = -\infty$
  is similar to the previous case, but replacing bigger by smaller.

Example 7.1.1:

(a)

$$\lim_{x \to 0} \frac{1}{x^2} = \infty.$$

(b)

$$\lim_{x \to \infty} \frac{1}{x} = 0.$$

(c)

$$\lim_{x \to \infty} \frac{x}{x+1} = 1.$$

Warning: Not every limit exists!

Example 7.1.2: (a)

$$\lim_{x \to \infty} \sin x$$

does not exist as the function does not tend to a single value.

(b)

$$\lim_{x \to 0} \left(\frac{1}{x} \right)$$

does not exist as $\frac{1}{x}$ is very large in modulus and positive if $x > 0$ is very small, and is very large in modulus and negative if $x < 0$ is very small.

If the limits of f and g as x tends to c exist and are finite then we have:

$$\lim_{x\to c}(f(x) \pm g(x)) = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x).$$

$$\lim_{x\to c}(f(x)g(x)) = \left(\lim_{x\to c} f(x)\right)\left(\lim_{x\to c} g(x)\right).$$

$$\lim_{x\to c}(f(x)^n) = \left(\lim_{x\to c} f(x)\right)^n$$

for $n \in \mathbb{N}$.

$$\lim_{x\to c}\left(\frac{f(x)}{g(x)}\right) = \left(\frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}\right)$$

provided that $\lim_{x\to c} g(x) \neq 0$.

Example 7.1.3:

$$\lim_{x\to 2}\left((x+1)(x+2)^4\right) = \lim_{x\to 2}(x+1)\left(\lim_{x\to 2}(x+2)\right)^2 = 3 \times 4^4 = 768.$$

Warning: We cannot use the above identities if the limits of f or g do not exist, or are $\pm\infty$.

As Example 7.1.3 should suggest, if f is continuous at $x = c$ then

$$\lim_{x\to c} f(x) = f(c).$$

However, the usefulness of limits is that they can be used to investigate the properties of functions near values where they may not be defined. Looking at the asymptotes to a given function is also related to considering limits.

7.2 Limits of quotients of functions

A common situation where limits arise is the case where we quotient one function by another, particularly if the denominator equals zero at the point of interest. We have various methods for tackling such limits (provided that they exist).

If $f(c)$ and $g(c)$ exist and $g(c) \neq 0$ then we saw above that

$$\lim_{x \to c} \left(\frac{f(x)}{g(x)} \right) = \frac{f(c)}{g(c)}.$$

Example 7.2.1:

$$\lim_{x \to 1} \left(\frac{x^2 + 2x + 3}{x^2 - 7} \right) = \frac{6}{-6} = -1.$$

If $f(c) = g(c) = 0$ and f and g are polynomials then we can try to factorise.

Example 7.2.2: (a)

$$\lim_{x \to 1} \left(\frac{x^2 - 2x + 1}{x^2 - 3x + 2} \right) = \lim_{x \to 1} \left(\frac{(x-1)(x-1)}{(x-1)(x-2)} \right) = \frac{0}{-1} = 0.$$

(b)

$$\lim_{x \to 1} \left(\frac{x^3 - 1}{x^3 + 2x^2 - x - 2} \right) = \lim_{x \to 1} \left(\frac{(x-1)(x^2 + x + 1)}{(x-1)(x^2 + 3x + 2)} \right) = \frac{3}{6} = \frac{1}{2}.$$

Similar methods in reverse may work for other quotients.

Example 7.2.3:

$$\lim_{x\to 4}\left(\frac{\sqrt{x}-2}{x-4}\right) = \lim_{x\to 4}\left(\frac{(\sqrt{x}-2)(\sqrt{x}+2)}{(x-4)(\sqrt{x}+2)}\right)$$
$$= \lim_{x\to 4}\left(\frac{x-4}{(x-4)(\sqrt{x}+2)}\right) = \lim_{x\to 4}\left(\frac{1}{\sqrt{x}+2}\right) = \frac{1}{4}.$$

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For $\lim_{x\to 0}$ and $\lim_{x\to\infty}$ of a quotient of polynomials we have the following methods. If we want $\lim_{x\to 0}(\frac{f(x)}{g(x)})$ we first simplify until one or both of f and g has a constant term.

Example 7.2.4: (a)

$$\lim_{x\to 0}\left(\frac{ax^2+bx+c}{Ax^2+Bx+C}\right) = \frac{c}{C} \quad \text{if } C \neq 0.$$

(b)

$$\lim_{x\to 0}\left(\frac{4x^2+3x^3+x^7}{2x^2+x^5}\right) = \lim_{x\to 0}\left(\frac{4+3x+x^5}{2+x^3}\right) = \frac{4}{2} = 2$$

(c)

$$\lim_{x\to 0}\left(\frac{3x^2+4x^3+x^4}{x+x^7}\right) = \lim_{x\to 0}\left(\frac{3x+4x^2+x^3}{1+x^6}\right) = \frac{0}{1} = 0.$$

To determine $\lim_{x\to\infty}(\frac{f(x)}{g(x)})$ for polynomials f and g we replace x by $\frac{1}{y}$ and use $\lim_{y\to 0}$.

Example 7.2.5:

$$\lim_{x\to\infty}\left(\frac{3+4x+2x^2}{5-x-3x^2}\right) = \lim_{y\to 0}\left(\frac{3+4/y+2/y^2}{5-1/y-3/y^2}\right)$$
$$= \lim_{y\to 0}\left(\frac{3y^2+4y+2}{5y^2-y-3}\right) = -\frac{2}{3}$$

We have one more general rule which can be applied to quotients of arbitrary functions, called l'Hôpital's rule: If $f(c) = g(c) = 0$ and $\lim_{x\to c}(\frac{f'(x)}{g'(x)})$ exists then

$$\lim_{x\to c}\left(\frac{f(x)}{g(x)}\right) = \lim_{x\to c}\left(\frac{f'(x)}{g'(x)}\right).$$

This can also be applied to higher derivatives if $f'(c) = g'(c) = 0$.

Example 7.2.6: (a) Returning to example 7.2.2(b) we have

$$\lim_{x\to 1}\left(\frac{x^3-1}{x^3+2x^2-x-2}\right) = \lim_{x\to 1}\left(\frac{3x^2}{3x^2+4x-1}\right) = \frac{3}{6} = \frac{1}{2}.$$

(b)

$$\lim_{x\to 0}\left(\frac{\sin(x)}{x}\right) = \lim_{x\to 0}\left(\frac{\cos(x)}{1}\right) = \frac{1}{1} = 1.$$

This latter example is a standard limit, and should be learnt. Two other standard limits:

$$\lim_{x\to 0}(x^k \ln x) = 0 \qquad \text{for } k > 0$$

$$\lim_{x\to\infty}(x^k e^{-x}) = 0 \qquad \text{for } k > 0$$

7.3 Convergent series and power series

We say that a sequence of numbers $a_1, a_2, \ldots, a_n, \ldots$ converges to the number L if the terms in the sequence become arbitrarily close to L for n sufficiently large. In this case we write

$$\lim_{n \to \infty} a_n = L.$$

For example the sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$ converges to 0 while the sequence $-1, 1, -1, 1, \ldots, (-1)^n, \ldots$ does not converge. As before, this rather vague definition can be made more precise.

Given a sequence of numbers $a_1, a_2, \ldots, a_n, \ldots$ we define the nth partial sum S_n to be

$$S_n = \sum_{i=1}^{n} a_i.$$

We say that the sum $\sum_{i=1}^{\infty} a_i$ converges with sum S if $\lim_{n \to \infty} S_n = S$.

Example 7.3.1: Consider the geometric series with initial value a and constant ratio r. This has nth partial sum

$$S_n = a + ar + \cdots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1}.$$

The sum $\sum_{k=1}^{\infty} ar^{k-1}$ exists if $\lim_{n \to \infty} S_n$ exists.

If $|r| > 1$ then $|r^n|$ gets larger and larger, so there is no limit. But if $|r| < 1$ then r^n tends to zero as n tends to infinity. Hence

$$\lim_{n \to \infty} \left(\frac{a(r^n - 1)}{r - 1} \right) = \frac{a}{1 - r}$$

if $|r| < 1$. If $|r| = 1$ then it can be shown that there is no limit.

In summary, we have

$$\sum_{k=1}^{\infty} ar^{k-1} = \begin{cases} \frac{a}{1-r} & \text{if } |r| < 1 \\ \text{does not exist} & \text{if } |r| \geq 1. \end{cases}$$

This result must be known — usually it is remembered in the form

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

provided that $|x| < 1$.

This is an example of a power series: an infinite sum of increasing powers of a variable x. Typically such series converge for some values of x but not for others. Next we will consider two special (and related) power series: Taylor and Maclaurin series.

7.4 Taylor series

Suppose that we have a function which can be differentiated at least n times on some interval containing a point c. We might like to approximate this function by a polynomial (because this is easier to work with), at least in the region near to c.

So what whould be a good approximation to take? It is reasonable to require that the approximation (which we will denote by $p_n(x)$) should agree with $f(x)$ at the point $x = c$, and that the functions should have the same first, second, ..., nth derivatives at the point c. That is, that

$$p_n^{(i)}(c) = f^{(i)}(c)$$

for $0 \leq i \leq n$.

Consider the function

$$p_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

Clearly $p_n(c) = f(c)$, and it is easy to check that

$$p_n{}^{(i)}(c) = f^{(i)}(c)$$

for $1 \leq i \leq n$. Thus $p_n(x)$ approximates $f(x)$ in the desired manner.

We define the Taylor series of f about c to be the infinite sum

$$T(f,c) = \sum_{i \geq 0} \frac{f^{(i)}(c)}{i!}(x-c)^i$$

where $f^{(0)}(x) = f(x)$ and $0! = 1$. When $c = 0$ this is called the Maclaurin series of f.

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Example 7.4.1: Find the Maclaurin series of $f(x) = e^x$.

We have $f'(x) = e^x = f''(x) = \ldots$ for all x, and so $f^{(n)}(0) = 1$ for all n. Thus the Maclaurin series for e^x is

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

As yet we have no guarantee that the Taylor series for a given function will actually converge to equal that function. Indeed, in general it will not converge to the correct value at every value of x. In this course we will not investigate the general problem of convergence, but instead look at some important examples and state (without proof) when they converge.

The following examples should be memorised.

Function	Series	General term	Converges
e^x	$1 + x + \frac{x^2}{2!} + \cdots$	$\frac{x^n}{n!}$	all x
$\sin x$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	$\frac{(-1)^n x^{2n+1}}{(2n+1)!}$	all x
$\cos x$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	$\frac{(-1)^n x^{2n}}{(2n)!}$	all x
$\ln(1+x)$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$	$\frac{(-1)^{n+1} x^n}{n}$	$-1 < x \leq 1$

Note that we could not give the expansion of $\ln x$ about zero in the above list, but instead about one. Also, when using the formulas for cos and sin we must use radians.

We can also extend the binomial theorem for all real powers p:

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \cdots + \frac{p(p-1)\cdots(p+1-n)}{n!}x^n + \cdots$$

for $-1 < x < 1$.

We can often get one power series by modifying another.

Example 7.4.2: Find a series for $\ln(2+3x)$ and state its region of convergence.

$$\ln(2+3x) = \ln\left(2\left(1+\frac{3x}{2}\right)\right) = \ln 2 + \ln\left(1+\frac{3x}{2}\right).$$

Using the sequence for $\ln(1+u)$ with $u = \frac{3x}{2}$ we have

$$\ln(2+3x) = \ln 2 + \sum_{n\geq 1} \frac{(-1)^{n+1}}{n} \left(\frac{3x}{2}\right)^n .$$

This sequence converges when $-1 < u \leq 1$, i.e. $-\frac{2}{3} < x \leq \frac{2}{3}$.

Example 7.4.3: Find a series for $f(x) = \cos^2 x$.

$$\begin{aligned} \cos^2 x &= \tfrac{1}{2} + \tfrac{1}{2}\cos 2x \\ &= \tfrac{1}{2} + \tfrac{1}{2}\left[1 - \tfrac{(2x)^2}{2!} + \tfrac{(2x)^4}{4!} - \tfrac{(2x)^6}{6!} + \cdots\right] \\ &= 1 - x^2 + \tfrac{x^4}{3} - \tfrac{2x^6}{45} + \cdots \end{aligned}$$

valid for all x.

We often use Taylor series methods to approximate functions close to a value c by a polynomial.

Example 7.4.4: When x is small we have

$$\sin x \approx x \qquad \text{and} \qquad \cos x \approx 1 - \frac{x^2}{2}.$$

These are known as the small angle approximations, and should be known.

Example 7.4.5: Express $\cosh x$ as a series of powers up to the term in x^6. Hence show that, near 0,

$$\cosh x \approx 1 + \frac{x^2}{2}.$$

$$\begin{aligned}
\cosh x &= \frac{1}{2}(e^x + e^{-x}) \\
&= \frac{1}{2}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + 1 - x + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \cdots\right) \\
&= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots
\end{aligned}$$

for all real x. Near 0 the terms in x^4 and higher contribute negligibly, so we obtain the desired approximation.

Taking this approach to its logical conclusion we see that series expansions are a useful tool for calculating limits of functions.

Example 7.4.6: Calculate

$$\lim_{x \to 0}\left(\frac{\sin x}{x}\right).$$

We have

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

and so

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$

Now

$$\lim_{x \to 0}\left(\frac{\sin x}{x}\right) = \lim_{x \to 0}\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots\right) = 1.$$

Example 7.4.7: Calculate

$$\lim_{x\to 0}\left(\frac{e^{2x}-e^{x}}{x}\right).$$

We have

$$\begin{aligned} e^{2x}-e^{x} &= 1+2x+\frac{(2x)^2}{2!}+\frac{(2x)^3}{3!}+\cdots-\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right) \\ &= x+\frac{3x^2}{2!}+\frac{7x^3}{3!}+\cdots \end{aligned}$$

and so

$$\lim_{x\to 0}\left(\frac{e^{2x}-e^{x}}{x}\right)=\lim_{x\to 0}\left(1+\frac{3x}{2!}+\frac{7x^2}{3!}+\cdots\right)=1.$$

In all of the examples in this section we have concentrated on Maclaurin's series: the special case $c=0$. This was just to make the calculations easier to write down. For general values of c the methods are the same.

Example 7.4.8: Obtain the Taylor's expansion of $x^2\ln x$ in powers of $(x-1)$ up to $(x-1)^4$.

Let $f(x)=x^2\ln x$. Then we want

$$p_4(x)=f(1)+f'(1)(x-1)+\frac{f''(1)}{2!}(x-1)^2+\cdots+\frac{f^{(4)}(1)}{4!}(x-1)^4.$$

You should check that this gives

$$(x-1)+\frac{3}{2}(x-1)^2+\frac{1}{3}(x-1)^3-\frac{1}{12}(x-1)^4.$$

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Sometimes it is useful to use the Leibnitz rule, which gives a formula for the *n*th derivative of a product of functions.

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(fg) = \frac{\mathrm{d}^n}{\mathrm{d}x^n}(f)g + \binom{n}{1}\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}(f)\frac{\mathrm{d}}{\mathrm{d}x}(g) + \binom{n}{2}\frac{\mathrm{d}^{n-2}}{\mathrm{d}x^{n-2}}(f)\frac{\mathrm{d}^2}{\mathrm{d}x^2}(g) + \cdots$$
$$\cdots + \binom{n}{n-1}\frac{\mathrm{d}}{\mathrm{d}x}(f)\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}(g) + f\frac{\mathrm{d}^n y}{\mathrm{d}x^n}(g).$$

So for example

$$\frac{\mathrm{d}^3}{\mathrm{d}x^3}(fg) = \frac{\mathrm{d}^3}{\mathrm{d}x^3}(f)g + 3\frac{\mathrm{d}^2}{\mathrm{d}x^2}(f)\frac{\mathrm{d}}{\mathrm{d}x}(g) + 3\frac{\mathrm{d}}{\mathrm{d}x}(f)\frac{\mathrm{d}^2}{\mathrm{d}x^2}(g) + f\frac{\mathrm{d}^3}{\mathrm{d}x^3}(g).$$

As can be seen, this is very similar to the binomial theorem, and can also be proved by induction (using the product rule).

7.5 The formal definition of a limit

Our discussion of limits has been somewhat unsatisfactory, as we have not had a rigorous definition of a limit to work with. In this section we will briefly explain how to make this more precise. To give a detailed examination of limits is beyond the scope of this module, so we will restrict ourselves to the definition and some basic examples.

We say that the limit of f as x tends to c is L, written

$$\lim_{x \to c} f(x) = L$$

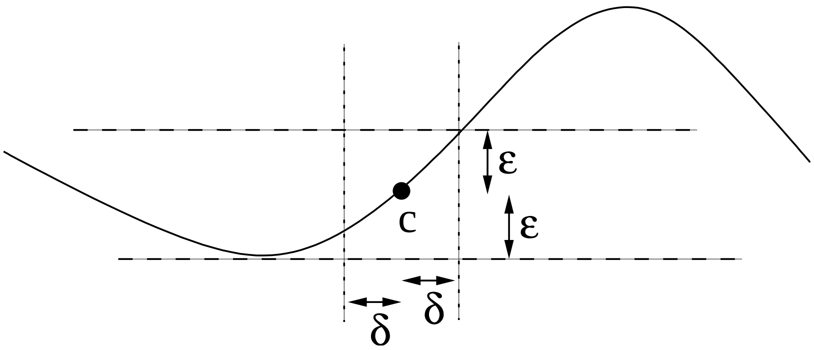
if for all $\epsilon > 0$ there exists some $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - c| < \delta$.

Roughly, this says that $f(x)$ will always be as close to L as we like, provided that we choose x to be sufficiently close to c.

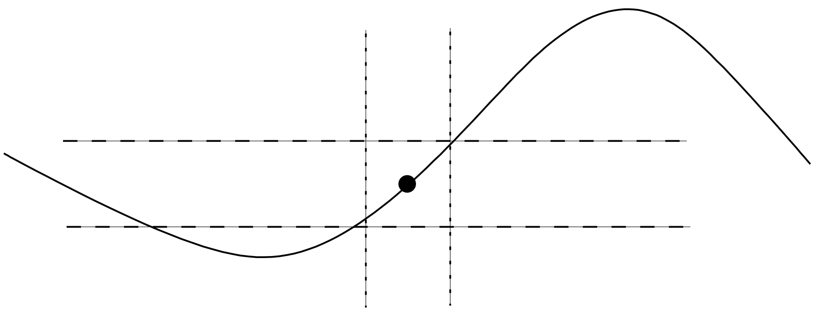
Note that the choice of δ will depend on ϵ.

Sometimes this definition is explained as though we were playing a game. One player chooses a positive number ϵ, and the second player then has to choose a second positive number δ so that $f(x)$ is always within ϵ of L if x is within δ of c. If the second player can always do this, then the function has limit L at c.

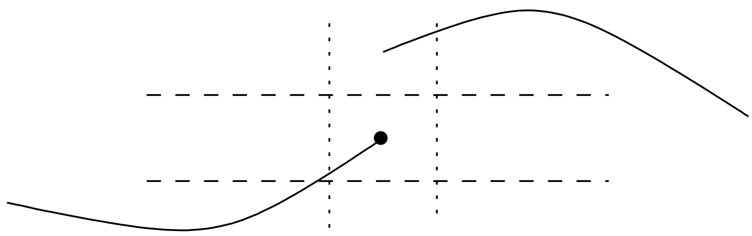
For example, in the following picture Player 1 picks the horizontal strip shown. Then Player 2 can pick the vertical strip as indicated to satisfy the condition.



If Player 1 chooses a narrower horizontal strip as in the next picture, Player 2 can still choose a vertical strip as shown.



For an example where the limit does not exist, consider the function in the following figure.



Here we have indicated a horizontal strip for which no vertical strip will ever guarantee that the curve between the vertical lines will always lie inside the horizontal strip.

Thus this is an example where the limit condition is not satisfied.

These pictures may help us to understand the definition, but they do not help us to apply it. The best way to see how to do this is through some examples.

Example 7.5.1: Show that

$$\lim_{x \to 3} 2x = 6.$$

Here $f(x) = 2x$, $c = 3$, and $L = 6$. Given $\epsilon > 0$, we must find $\delta > 0$ such that

$$|f(x) - 6| < \epsilon$$

whenever $0 < |x - 3| < \delta$. But if we choose $\delta = \epsilon/2$ then for all $0 < |x - 3| < \delta$ we have

$$|f(x) - 6| = |2x - 6| = 2|x - 3| < 2\delta = \epsilon$$

as required.

Example 7.5.2:

Show that

$$\lim_{x \to 3} x^2 = 9.$$

Here $f(x) = x^2$, $c = 3$ and $L = 9$. Given $\epsilon > 0$, we must find $\delta > 0$ such that

$$|f(x) - 9| < \epsilon$$

whenever $0 < |x - 3| < \delta$. It is now not as easy to see how to choose δ.

First we suppose that we have picked δ, and see what happens to the equations.

If $|x - 3| < \delta$ then note that

$$|x + 3| = |x - 3 + 6| \leq |x - 3| + 6 \leq \delta + 6.$$

Therefore

$$|x^2 - 9| = |(x - 3)||(x + 3)| \leq \delta(\delta + 6) = \delta^2 + 6\delta.$$

If $\epsilon \geq 1$ then we can choose $\delta = 0.1$, and then

$$|x^2 - 9| < 0.1^2 + 0.6 < \epsilon$$

as required. If $\epsilon < 1$ then $\epsilon^2 < \epsilon$, and we can choose $\delta = \frac{\epsilon}{12}$. Then

$$|x^2 - 9| \leq \delta^2 + 6\delta \leq \frac{\epsilon^2}{144} + \frac{\epsilon}{2} \leq \epsilon(\frac{1}{144} + \frac{1}{2}) < \epsilon$$

as required. Thus we have shown that $\lim_{x \to 3} x^2 = 9$.