Mathematics for Actuarial Science (AS1051)

Coursework 1 (Solutions and marking scheme)

This is an assessed coursework, and will count towards your final grade. Solutions should be handed in to the SEMS general office (C108) by the stated deadline. Late submissions will be penalised.

DEADLINE: Monday 03/12/2012 at 16:00

1) *i)* We convert [5 marks]

$$\sqrt{12}\cos^2\theta - \sin 2\theta = 0$$

into

$$2\sqrt{3}\cos^2\theta - 2\cos\theta\sin\theta = 2\cos\theta\left(\sqrt{3}\cos\theta - \sin\theta\right) = 0.$$

This means we have a solution whenever

$$\cos\theta = 0 \qquad \text{or} \qquad \tan\theta = \sqrt{3},$$

that is

$$\theta = \pm\frac{\pi}{2} + 2n\pi \qquad \text{or} \qquad \theta = \frac{\pi}{3} + n\pi \qquad \text{with } n \in \mathbb{Z}.$$

ii) We use the identity $\log_a c = \log_b c / \log_b a$ to re-write [5 marks]

$$\log_3(x) = \log_{27}(\left|2x^2 + 5x - 6\right|) = \frac{\log_3(\left|2x^2 + 5x - 6\right|)}{\log_3 27}.$$

Next we use $\log_3 27 = \log_3 3^3 = 3\log_3 3 = 3$ and $3\log_3(x) = \log_3(x^3)$, such that

$$\log_3(x^3) = \log_3(\left|2x^2 + 5x - 6\right|) \quad \Rightarrow \quad x^3 = \left|2x^2 + 5x - 6\right|.$$

Thus

$$\text{for } 2x^2 + 5x - 6 > 0: \quad x^3 - 2x^2 - 5x + 6 = 0 \quad \Rightarrow \quad x_1 = 1,\ x_2 = 3,\ x_3 = -2$$
$$\text{for } 2x^2 + 5x - 6 < 0: \quad x^3 + 2x^2 + 5x - 6 = 0 \quad \Rightarrow \quad \text{2 complex, 1 real solution}$$

Since $\log_3(x)$ is only defined for $x > 0$ we have to discard the solution x_3. However, $x_1 = 1$ and $x_2 = 3$ are two viable solutions. (The real solution of $x^3 + 2x^2 + 5x - 6 = 0$ is positive and therefore also viable, although more complicated, i.e. $\frac{1}{3}\left[-2 - \frac{11}{\sqrt[3]{118+3\sqrt{1695}}} + \sqrt[3]{118 + 3\sqrt{1695}}\right] \approx 0.8204$. Only two solutions are needed to get full marks.)

$\boxed{\sum = 10}$

2) Integrate

$$I_n := \int \cos^n \theta d\theta,$$

by parts with $u = \cos^{n-1}\theta$ and $dv/d\theta = \cos\theta$. Therefore $du/d\theta = (1-n)\cos^{n-2}\theta \sin\theta$ and $v = \sin\theta$, such that

$$\begin{aligned} I_n &= \sin\theta\cos^{n-1}\theta + (n-1)\int \cos^{n-2}\theta \sin^2\theta d\theta, \\ &= \sin\theta\cos^{n-1}\theta + (n-1)\int \cos^{n-2}\theta \left(1-\cos^2\theta\right) d\theta, \\ &= \sin\theta\cos^{n-1}\theta + (n-1)(I_{n-2} - I_n). \end{aligned}$$

Therefore we find the recursive equation [2 marks]

$$I_n = \frac{1}{n}\sin\theta\cos^{n-1}\theta + \frac{n-1}{n} I_{n-2}$$

In order to find I_6 we compute first $I_0 = \int d\theta = \theta$. Then we have [4 marks]

$$\begin{aligned} I_2 &= \frac{1}{2}\sin\theta\cos\theta + \frac{1}{2}I_0 = \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta, \\ I_4 &= \frac{1}{4}\sin\theta\cos^3\theta + \frac{3}{4}I_2 = \frac{1}{4}\sin\theta\cos^3\theta + \frac{3}{4}(\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta), \\ &= \frac{3}{8}\theta + \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta, \\ I_6 &= \frac{1}{6}\sin\theta\cos^5\theta + \frac{5}{6}I_4 = \frac{1}{6}\sin\theta\cos^5\theta + \frac{5}{6}(\frac{3}{8}\theta + \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta), \\ &= \frac{5}{16}\theta + \frac{15}{64}\sin 2\theta + \frac{3}{64}\sin 4\theta + \frac{1}{192}\sin 6\theta. \end{aligned}$$

To find I_7 we compute first $I_1 = \int \cos\theta d\theta = \sin\theta$. Then we have [4 marks]

$$\begin{aligned} I_3 &= \frac{1}{3}\sin\theta\cos^2\theta + \frac{2}{3}I_1 = \frac{1}{3}\sin\theta\cos^2\theta + \frac{2}{3}\sin\theta, \\ I_5 &= \frac{1}{5}\sin\theta\cos^4\theta + \frac{4}{5}I_3 = \frac{1}{5}\sin\theta\cos^4\theta + \frac{4}{5}(\frac{1}{3}\sin\theta\cos^2\theta + \frac{2}{3}\sin\theta), \\ &= \frac{5}{48}\sin 3\theta + \frac{1}{80}\sin 5\theta + \frac{5}{8}\sin\theta, \\ I_7 &= \frac{1}{7}\sin\theta\cos^6\theta + \frac{6}{7}I_5 = \frac{1}{7}\sin\theta\cos^6\theta + \frac{6}{7}(\frac{5}{48}\sin 3\theta + \frac{1}{80}\sin 5\theta + \frac{5}{8}\sin\theta), \\ &= \frac{7}{64}\sin 3\theta + \frac{7}{320}\sin 5\theta + \frac{1}{448}\sin 7\theta + \frac{35}{64}\sin\theta. \end{aligned}$$

$\boxed{\sum = 10}$

3) First we assume [4 marks]

$$\frac{3x+15}{x-2} > 0 \quad \Rightarrow \quad x > 2 \quad \text{or} \quad x < -5 \qquad \text{(case 1)}$$

Then the inequality becomes

$$\frac{3x+15}{x-2} - \frac{2x+1}{x-4} = \frac{x^2+6x-58}{(x-2)(x-4)} = \frac{(x+3-\sqrt{67})(x+3+\sqrt{67})}{(x-2)(x-4)} < 0.$$

This is true if either one or three of the factors on the left hand side are negative, i.e.

$$-3-\sqrt{67} < x < 2 \quad \text{or} \quad 4 < x < -3+\sqrt{67}.$$

Combining this with (case 1) gives

$$-3-\sqrt{67} < x < -5 \quad \text{or} \quad 4 < x < -3+\sqrt{67}. \qquad \text{(sol case 1)}$$

Next we assume [4 marks]

$$\frac{3x+15}{x-2} < 0 \quad \Rightarrow \quad -5 < x < 2 \qquad \text{(case 2)}$$

Then the inequality becomes

$$-\frac{3x+15}{x-2} - \frac{2x+1}{x-4} = \frac{62-5x^2}{(x-2)(x-4)} = \frac{(x-\sqrt{62/5})(x=\sqrt{62/5})}{(x-2)(x-4)} < 0.$$

This is true if either one or three of the factors on the left hand side are negative, i.e.

$$x < -\sqrt{\frac{62}{5}} \quad \text{or} \quad 2 < x < \sqrt{\frac{62}{5}} \quad \text{or} \quad x > 4.$$

Combining this with (case 2) gives

$$-5 < x < -\sqrt{\frac{62}{5}}. \qquad \text{(sol case 2)}$$

The overall solution is the union of (sol case 1) and (sol case 2), that is

$$-3-\sqrt{67} < x < -\sqrt{\frac{62}{5}} \quad \text{or} \quad 4 < x < -3+\sqrt{67}.$$

$\boxed{\sum = 8}$

4) *i)* We have [9 marks]

$$\frac{6x^5+x^2+x+2}{(x^2+2x+1)(2x^2-x+4)(x+1)} = \frac{6x^5+x^2+x+2}{(2x^2-x+4)(x+1)^3}$$

This is not a proper fraction, so we write

$$\begin{aligned}\frac{6x^5+x^2+x+2}{(2x^2-x+4)(x+1)^3} &= 3 - \frac{15x^4+21x^3+32x^2+32x+10}{(x+1)^3(2x^2-x+4)} \\ &= 3 - \left(\frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} + \frac{Dx+E}{2x^2-x+4}\right)\end{aligned}$$

Therefore

$$\begin{aligned}15x^4+21x^3+32x^2+32x+10 = {} & 2Ax^4+3Ax^3+4Ax^2+7Ax+4A+2Bx^3 \\ & +Bx^2+3Bx+4B+2Cx^2-Cx+4C+Dx^4 \\ & +3Dx^3+3Dx^2+Dx+Ex^3+3Ex^2+3Ex \\ & +E\end{aligned}$$

Comparing coefficients gives

$$
\begin{aligned}
-2A - D + 15 &= 0, \\
-3A - 2B - 3D - E + 21 &= 0, \\
-4A - B - 2C - 3D - 3E + 32 &= 0, \\
-7A - 3B + C - D - 3E + 32 &= 0, \\
-4A - 4B - 4C - E + 10 &= 0.
\end{aligned}
$$

Solving these equations gives $A = \frac{1920}{343}, B = -\frac{183}{49}, C = \frac{4}{7}, D = \frac{1305}{343}, E = \frac{90}{343}$. Hence

$$f(x) = 3 - \frac{1920}{343(x+1)} + \frac{183}{49(x+1)^2} - \frac{4}{7(x+1)^3} - \frac{45(29x+2)}{343\,(2x^2 - x + 4)}$$

ii) Therefore [3 marks]

$$
\begin{aligned}
f'(x) = {} & \frac{45(4x-1)(29x+2)}{343\,(2x^2-x+4)^2} - \frac{1305}{343\,(2x^2-x+4)} + \frac{1920}{343(x+1)^2} - \frac{366}{49(x+1)^3} \\
& + \frac{12}{7(x+1)^4}
\end{aligned}
$$

$$f'(0) = -\frac{9}{8}, \qquad f'(1) = \frac{3}{10}.$$

iii) We find [4 marks]

$$
\begin{aligned}
\frac{1}{5}\int f(x)dx = {} & \frac{3}{5} + \frac{3x}{5} - \frac{183}{245(x+1)} + \frac{2}{35(x+1)^2} - \frac{333 \arctan\left(\frac{4x-1}{\sqrt{31}}\right)}{686\sqrt{31}} \\
& - \frac{261 \log\left(2x^2 - x + 4\right)}{1372} - \frac{384}{343}\log(x+1)
\end{aligned}
$$

$\boxed{\sum = 16}$

5) We use the formula from the lecture

$$\arctan a + \arctan b = \arctan\left(\frac{a+b}{1-ab}\right) + p\pi,$$

where

$$p = \begin{cases} 1 & \text{for } -\pi < \arctan a + \arctan b < -\frac{\pi}{2} \\ 0 & \text{for } -\frac{\pi}{2} < \arctan a + \arctan b < \frac{\pi}{2} \\ -1 & \text{for } \frac{\pi}{2} < \arctan a + \arctan b < \pi \end{cases}.$$

Using $0 \leq \arctan(x) \leq \pi/4$ for $0 < x < 1$, $\pi/4 \leq \arctan(x) \leq \pi/2$ for $x > 1$ we have

$$\frac{\pi}{4} \leq \arctan\left(\frac{3}{2}\right), \arctan\left(\frac{5}{4}\right) \leq \frac{\pi}{2} \;\Rightarrow\; \frac{\pi}{2} \leq \arctan\left(\frac{3}{2}\right) + \arctan\left(\frac{5}{4}\right) \leq \pi$$

such that

$$\arctan\left(\frac{3}{2}\right) + \arctan\left(\frac{5}{4}\right) = \arctan\left(-\frac{22}{7}\right) + \pi.$$

Using the previous estimates and the fact that $\arctan(x)$ is odd we have

$$-\frac{\pi}{2} \leq \arctan\left(-\frac{5}{2}\right), \arctan\left(-\frac{8}{3}\right) \leq -\frac{\pi}{4} \Rightarrow -\pi \leq \arctan\left(-\frac{5}{2}\right)+\arctan\left(-\frac{8}{3}\right) \leq \frac{\pi}{2}$$

such that

$$\arctan\left(-\frac{5}{2}\right) + \arctan\left(-\frac{8}{3}\right) = \arctan\left(\frac{31}{34}\right) - \pi$$

With

$$-\frac{\pi}{2} \leq \arctan\left(-\frac{22}{7}\right) \leq -\frac{\pi}{4}, 0 \leq \arctan\left(\frac{31}{34}\right) \leq \frac{\pi}{2}$$
$$\Rightarrow -\frac{\pi}{2} \leq \arctan\left(-\frac{22}{7}\right) + \arctan\left(\frac{31}{34}\right) \leq \frac{\pi}{2}$$

$$\arctan\left(-\frac{22}{7}\right) + \arctan\left(\frac{31}{34}\right) = \arctan\left(-\frac{531}{920}\right),$$

such that

$$x = -\frac{531}{920}.$$

$$\boxed{\sum = 6}$$