Mathematics for Actuarial Science (AS1051)

2010 (Solutions)

Full marks can be obtained by answering all six questions.
All necessary working must be shown.

TIME ALLOWED: 90 minutes

1) The general term in the expansion is [3 marks]

$$\left(x^5 + \frac{2}{x^3}\right)^{13} \sim \binom{13}{k} \left(x^5\right)^{13-k} \left(\frac{2}{x^3}\right)^k$$
$$= \binom{13}{k} 2^k x^{5(13-k)-3k}$$

For this to be proportional to x we require $5(13-k) - 3k = 1$ and therefore $k = 8$. Then the coefficient becomes [3 marks]

$$\binom{13}{8} 2^8 = \frac{13!}{8!(13-8)!} 2^8 = \frac{9\ 10\ 11\ 12\ 13}{2\ 3\ 4\ 5} 2^8 = 2^8 \times 3^2 \times 11 \times 13.$$

$\boxed{\sum = 6}$

2) Completing the squares gives

$$2(x^2 - 2x + 1) - 2 + 3\left(y^2 + \frac{5}{3}y + \left(\frac{5}{6}\right)^2\right) - \left(\frac{5}{6}\right)^2 3 + 4 = 0$$
$$2(x-1)^2 + 3\left(y + \frac{5}{6}\right)^2 = 2 - 4 + \frac{25}{12} = \frac{1}{12}$$

The normal form of the ellipse is therefore

$$\frac{(x-1)^2}{1/24} + \frac{\left(y + \frac{5}{6}\right)^2}{1/36} = 1 \ ,$$

such that $a^2 = 1/24$ and $b^2 = 1/36$. [4 mark]

$\Rightarrow$ The centre is at $(1, -5/6)$. [1 mark]

$\Rightarrow$ The length of the major axis is $2a = 2/\sqrt{24} = 1/\sqrt{6}$. [1 mark]

$\Rightarrow$ The length of the minor axis is $2b = 2/\sqrt{36} = 1/3$.

$\Rightarrow$ The eccentricity is $e = \sqrt{1 - b^2/a^2} = \sqrt{1 - 24/36} = \sqrt{1 - 2/3} = 1/\sqrt{3}$. [1 mark]

$\Rightarrow$ The foci are at $(1 \pm ae, -5/6) = (1 \pm 1/6\sqrt{2}, -5/6)$. [1 mark]

$\Rightarrow$ The equation of the directrix is $x = 1 - b^2/ae = 1 - \sqrt{2}/6$. [1 mark]

$\boxed{\sum = 9}$

3) For

$$x(t) = 4t^2 + 3 \qquad \text{and} \qquad y(t) = \ln(2t^2 + 1)$$

we compute

$$\frac{dx}{dt} = 8t \qquad \text{and} \qquad \frac{dy}{dt} = \frac{4t}{2t^2 + 1}$$

and subsequently [2 marks]

$$\frac{d^2x}{dt^2} = 8 \qquad \text{and} \qquad \frac{d^2y}{dt^2} = -\frac{4(1 - 2t^2)}{(2t^2 + 1)^2}.$$

Therefore [3 marks]

$$\frac{dy}{dx} = \frac{1}{2(1 + 2t^2)}, \qquad \frac{d^2y}{dt^2}\frac{d^2t}{dx^2} = \frac{1 - 2t^2}{2(2t^2 + 1)^2}$$

and

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = \frac{d}{dt}\left(\frac{1}{2(1 + 2t^2)}\right)\frac{1}{8t}$$
$$= \frac{1}{4(2t^2 + 1)^2},$$

such that [4 marks]

$$p(t) = \frac{1}{2\,(2t^2 - 1)}.$$

$\boxed{\sum = 9}$

4) First show the validity for $n = 1$ [1 mark]

$$\frac{dy}{dx} = a\cos(ax) = a\sin\left(ax + \frac{\pi}{2}\right).$$

Next assume the validity for $n - 1$ [2 marks]

$$\frac{d^{n-1}y}{dx^{n-1}} = a^{n-1}\sin\left[ax + \frac{(n-1)\pi}{2}\right].$$

Therefore [3 marks]

$$\frac{d^ny}{dx^n} = \frac{d}{dx}\frac{d^{n-1}y}{dx^{n-1}} = a^{n-1}a\cos\left[ax + \frac{(n-1)\pi}{2}\right]$$
$$= a^n\sin\left[ax + \frac{(n-1)\pi}{2} + \frac{\pi}{2}\right] = a^n\sin\left[ax + \frac{n\pi}{2}\right].$$

$\boxed{\sum = 6}$

5) *i)* Using $\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$ we have [4 marks]

$$\int \cos 5\theta \cos 4\theta d\theta = \frac{1}{2}\int \cos\theta + \cos 9\theta d\theta = \frac{1}{2}\sin\theta + \frac{1}{18}\sin 9\theta + C$$

ii) We integrate by parts [7 marks]

$$I = 9 \int x^2 e^{3x} dx$$

with $u = x^2$, $\frac{dv}{dx} = e^{3x}$ and $\frac{du}{dx} = 2x$, $v = \frac{1}{3}e^{3x}$

$$I = 9x^2 \frac{1}{3} e^{3x} - 9 \int 2x \frac{1}{3} e^{3x} dx.$$

Integrating again by parts with $u = x$, $\frac{dv}{dx} = e^{3x}$ and $\frac{du}{dx} = 1$, $v = \frac{1}{3}e^{3x}$ gives

$$\begin{aligned} I &= 3x^2 e^{3x} - 9\frac{2}{3} x \frac{1}{3} e^{3x} + 9\frac{2}{3} \int \frac{1}{3} e^{3x} dx \\ &= 3x^2 e^{3x} - 2x e^{3x} + 2\frac{1}{3} e^{3x} + C = \frac{1}{3} e^{3x}(2 - 6x + 9x^2) + C. \end{aligned}$$

$\boxed{\sum = 11}$

6) We compute [7 marks]

$$\begin{aligned} f(0) &= 0, \\ f'(x) &= \cos(x)\log(x+1) + \frac{\sin(x)}{x+1} \quad \Rightarrow f'(0) = 0, \\ f''(x) &= \frac{2\cos(x)}{x+1} - \log(x+1)\sin(x) - \frac{\sin(x)}{(x+1)^2} \quad \Rightarrow f''(0) = 2, \\ f'''(x) &= -\log(x+1)\cos(x) - \frac{3\cos(x)}{(x+1)^2} - \frac{3\sin(x)}{x+1} + \frac{2\sin(x)}{(x+1)^3} \quad \Rightarrow f'''(0) = -3. \end{aligned}$$

Therefore the Taylor expansion up to the first two non-zero terms is [2 marks]

$$f(x) = \frac{2}{2}x^2 - \frac{3}{3!}x^3 = x^2 - \frac{1}{2}x^3.$$

$\boxed{\sum = 9}$