Mathematics for Actuarial Science (AS1051)

2011 (Solutions)

Full marks can be obtained by answering all six questions.
All necessary working must be shown.

TIME ALLOWED: 90 minutes

1) The solution is $w = 1$, $x = 4$, $y = -4$ and $z = 10$. $\boxed{\sum = 6}$

2) First we use the identity $\sinh^2\theta = \cosh^2\theta - 1$

$$7\cosh\theta - 2\sinh^2\theta = 8 \quad \Rightarrow \quad 8 - 7\cosh\theta + 2\cosh^2\theta - 2 = 0$$

therefore [3 marks]

$$(2\cosh\theta - 3)(\cosh\theta - 2) = 0$$

With $y = e^\theta$ solve [6 marks]

$$e^\theta + e^{-\theta} - 3 = 0 \quad \Leftrightarrow \quad y + y^{-1} - 3 = 0 \quad \Rightarrow \quad y_\pm = \frac{3 \pm \sqrt{5}}{2} \Rightarrow \quad \underline{x_{1/2} = \ln\left[\frac{3 \pm \sqrt{5}}{2}\right]}$$

$$\frac{1}{2}e^\theta + \frac{1}{2}e^{-\theta} - 2 = 0 \quad \Leftrightarrow \quad y + y^{-1} - 4 = 0 \quad \Rightarrow \quad y_\pm = 2 \pm \sqrt{3} \Rightarrow \quad \underline{x_{3,4} = \ln\left[2 \pm \sqrt{3}\right]}$$

$\boxed{\sum = 9}$

3) Take the transformation $(x, y) \to (X = x + 4, Y)$, then

$$x^2 + y^2 = (8 + x)^2 \quad -> \quad (X - 4)^2 + Y^2 = \ (X + 4)^2$$

Expanding this gives [2 marks]

$$Y^2 = \ 16X \quad \Rightarrow \quad \underline{a = 4}$$

To find $\mathcal{L}$ we use [2 marks]

$$m = \frac{Y_2 - Y_1}{X_2 - X_1} \quad \Rightarrow \quad m = \frac{7 - 1}{2 - 0} = 3 \ \Rightarrow \quad m = \frac{Y - 1}{X - 0} \Rightarrow \quad \underline{Y = 3X + 1}.$$

For the point of intersection with the parabola we need to solve [2 marks]

$$(3X + 1)^2 = 16X \Rightarrow \quad 9X^2 - 10X + 1 = 0 \Rightarrow \quad X = \frac{1}{9}, 1 \Rightarrow \quad Y = \frac{4}{3}, 4.$$

The points of intersection are therefore $\underline{P_1 = (\frac{1}{9}, \frac{4}{3})}$ and $\underline{P_2 = (1,4)}$. For the lines perpendicular to $\mathcal{L}$ we have [3 marks]

$$\mathcal{L}_1^\perp : \quad -\frac{1}{3} = \frac{Y - \frac{4}{3}}{X - \frac{1}{9}} \quad \Rightarrow \quad \underline{Y = -\frac{X}{3} + \frac{37}{27}}$$
$$\mathcal{L}_2^\perp : \quad -\frac{1}{3} = \frac{Y - 4}{X - 1} \quad \Rightarrow \quad \underline{Y = -\frac{X}{3} + \frac{13}{3}}.$$

$\boxed{\sum = 9}$

4) Multiply by $(x+2)^2$

$$(x+2)^2 \frac{(2 - 3x + x^2)}{(x+2)} < (x+2)^2 \quad \Rightarrow \quad (x-2)(2 - 3x + x^2 - x - 2) < 0.$$

Therefore [3 marks]

$$(x-2)(x-4)x < 0,$$

such that $\underline{x < -2}$ or $\underline{0 < x < 4}$. [3 marks]

$\boxed{\sum = 6}$

5) [11 marks] We compute

i) $4 \int \sin 5\theta \cos 9\theta d\theta = 2 \int \sin(14\theta) d\theta - 2 \int \sin(4\theta) d\theta = \frac{1}{2} \cos(4\theta) - \frac{1}{7} \cos(14\theta)$

[3 marks]

ii) $\int \frac{4x^2 + 9x + 7}{(3+x)(x^2+x+2)} dx = \int \frac{2}{3+x} dx + \int \frac{2x+1}{x^2+x+2} dx = 2 \ln|x+3| + \ln|x^2+x+2|$

[8 marks]

$\boxed{\sum = 11}$

6) We compute

$$\begin{aligned}
f(x) &= \arctan(1+x) \;\Rightarrow\; f(0) = \frac{\pi}{4}, \\
f'(x) &= \frac{1}{(x+1)^2 + 1} \;\Rightarrow\; f'(0) = \frac{1}{2}, \\
f''(x) &= -\frac{2(x+1)}{((x+1)^2+1)^2} \;\Rightarrow\; f''(0) = -\frac{1}{2}, \\
f'''(x) &= \frac{8(x+1)^2}{((x+1)^2+1)^3} - \frac{2}{((x+1)^2+1)^2} \;\Rightarrow\; f'''(0) = \frac{1}{2}.
\end{aligned}$$

Therefore the Taylor expansion up to the first two non-zero terms is [6 marks]

$$\begin{aligned}
f(x) &= f(0) + f'(0)x + \frac{f''(0)}{2} x^2 + \frac{f'''(0)}{3!} x^3 + \mathcal{O}\left(x^4\right) \\
&= \frac{\pi}{4} + \frac{x}{2} - \frac{x^2}{4} + \frac{x^3}{12} + \mathcal{O}\left(x^4\right).
\end{aligned}$$

[3 marks]

$\boxed{\sum = 9}$