Mathematics for Actuarial Science 4

1. Differentiate the following functions with respect to x:
   (a) $4x^3 - 2x^2 + 1 + x^{-2}$,
   (b) $\cos(3x + 2)$,
   (c) $\frac{x}{x^2+1}$,
   (d) $x \ln x$,
   (e) $\sqrt{1 + x^2}$,
   (f) $e^x \cos(x^2)$,
   (g) $\frac{x-8}{(x+2)(2x-1)}$,
   (h) x^{e^x},
   (i) $\cos(\tan(x^2))$,

2. Differentiate the following functions with respect to x:
   (a) $x^3 + 3xy^2 - 2xy + x^{-4}y^{-3} = 0$,
   (b) $\cos(x)\sin(y) = 1$,
   (c) $\cos(y\tan(x)) = \ln(xy)$,
   (d) $\frac{x+2\sin(y)+4}{(x-y)} = \cot(y)$,
   (e) $y\tan(x^{-1}) + x\sec(y^{-1}) = e^x$,

3. Differentiate the following functions with respect to x:
   (a) $x = t^2 + 2t + 1$, $y = 7t^3$,
   (b) $x = \cos(t)$, $y = \cot(t)$,
   (c) $x = \ln(\sqrt{1 + 2t^2})$, $y = t^3 + t$,
   (d) $x = \cos(t)e^{t^2}$, $y = \sin(t)e^{-t^2}$,
   (e) $x = \frac{1}{\ln(t)}$, $y = 3\tan(4t)$.

4. Given that $y = e^{-x}\sin(x\sqrt{3})$, prove that
$$\frac{\mathrm{d}y}{\mathrm{d}x} = -2e^{-x}\sin\left(x\sqrt{3} - \frac{\pi}{3}\right).$$
   Show also that $\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = ky$ for some constant k, and state the value of k.

5. If $f(x) = 3 - \frac{x^2}{4} + \ln\left(\frac{x}{2}\right)$,
   (a) show that there is a root α of $f(x) = 0$ such that $0.09 < \alpha < 0.1$
   (b) find $f'(x)$ and obtain the value of β such that $f'(\beta) = 0$.

6. $(*)$ If $y = \ln\left(1 + \sqrt{(1+x)}\right)$, show that

$$4x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} = (1+x)^{-\frac{3}{2}}.$$