Maths for Actuarial Science Coursewor 1: Answers

1. Easiest way is to sketch curves and inspect. Algebraically, any method is OK provided all cases are considered. Please deduct marks for each time a case is left out. One way: Let
$$A = \frac{x+1}{x-2} \quad \text{and} \quad B = \frac{x-1}{x+2}.$$
Equation equivalent to $A^2 < B^2$. Now multiply through and simplify.
The denominator equals $(x-2)^2(x+2)^2$ which is positive except at $x = \pm 2$ where the original equation is not defined.
The numerator simplifies to $x(x^2+2)$ and we require this to be negative.
As $x^2 + 2 > 0$, the solution is $x < 0$ with $x \neq -2$. **[Total: 10]**

2. (i) Centre of ellipse midway between foci, so at $(5, 2)$. **[1]**
Major axis has length $2a = 6$, so $a = 3$. **[1]**
Distance of foci from centre is $ae = 1$, so $e = \frac{1}{3}$. **[1]**
Then $b^2 = a^2(1 - e^2)$ implies that $b^2 = 8$. **[1]**
Thus the equation is
$$\frac{(x-5)^2}{9} + \frac{(y-2)^2}{8} = 1.$$
[1]
(ii) Any tangent line at a point X is perpendicular to a radius passing through X. Consider the right-angled triangle OXP where O is the centre of the circle. This has hypotenuse of length $|OP|$ and one side of length the circle radius, so the length of the third side (by Pythagoras) is independent of the choice of X. (No marks for doing an example only. A picture is sufficient if it is explained properly.) **[5]**
[Total: 10]

3. First note that
$$\sin 2\theta + \sin 3\theta = 2\sin\left(\frac{5\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) \quad \text{and } \sin 5\theta = 2\sin\left(\frac{5\theta}{2}\right)\cos\left(\frac{5\theta}{2}\right)$$
and hence
$$\sin 2\theta + \sin 3\theta + \sin 5\theta = 2\sin\left(\frac{5\theta}{2}\right)\left(\cos\left(\frac{\theta}{2}\right) + \cos\left(\frac{5\theta}{2}\right)\right).$$
This gives
$$\sin 2\theta + \sin 3\theta + \sin 5\theta = 4\sin\left(\frac{5\theta}{2}\right)\cos\left(\frac{3\theta}{2}\right)\cos\theta.$$
[5]
Thus the solutions are given by
$$\frac{5\theta}{2} = n\pi, \quad \frac{3\theta}{2} = \pm\frac{\pi}{2} + 2n\pi, \quad \theta = \pm\frac{\pi}{2} + 2n\pi$$

with $n \in \mathbb{Z}$. This gives

$$\theta = \frac{2}{5}n\pi, \quad \pm\frac{\pi}{3} + \frac{4}{3}n\pi, \quad \theta = \frac{\pi}{2} + n\pi$$

with $n \in \mathbb{Z}$ (or equivalent). (Please deduct a mark for approximate answers, and for not stating the domain of n.) **[5]**

[Total: 10]

4. (i)(a)

$$\frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} = \sec x.$$

[2]

(b)

$$3(\sin^2 x \cos x \cos 3x - \sin^3 x \sin 3x - \cos^2 x \sin x \sin 3x + \cos^3 x \cos 3x) = 3\cos 4x.$$

[2]

(c) $e^{x \ln a} \ln a = a^x \ln a$. **[2]**

(ii) $\frac{\mathrm{d}x}{\mathrm{d}t} = 8t - 5$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = 3t^2 - 2t + 1$. Hence $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}(\frac{\mathrm{d}x}{\mathrm{d}t})^{-1} = \frac{3t^2-2t+1}{8t-5}$. **[2]**

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{3t^2 - 2t + 1}{8t - 5}\right)\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{24t^2 - 30t + 2}{(8t-5)^3}.$$

[2]

[Total: 10]

5. (a)The fraction simplifies to

$$1 - \frac{x^2 - 9x - 9}{(x^2 - 9)(x + 1)}.$$

[1]

Let

$$\frac{x^2 - 9x - 9}{(x^2 - 9)(x + 1)} = \frac{A}{x - 3} + \frac{B}{x + 3} + \frac{C}{x + 1}.$$

[1]

Solve to find $A = -\frac{9}{8}$, $B = \frac{9}{4}$, $C = -\frac{1}{8}$, and hence

$$\frac{x^3}{(x^2 - 9)(x + 1)} = 1 + \frac{1}{8}\left(\frac{9}{x - 3} + \frac{1}{x + 1} - \frac{18}{x + 3}\right).$$

[2]

Therefore the integral equals

$$x + \frac{1}{8}\left(9\ln(x - 3) + \ln(x + 1) - 18\ln(x + 3)\right) + C.$$

[1]

(b)

$$\begin{aligned}\int x^2 e^{-2x}\,dx &= -\frac{1}{2}x^2e^{-2x} + \int xe^{-2x}\,dx \\ &= -\frac{1}{2}x^2e^{-2x} - \frac{1}{2}xe^{-2x} + \int \frac{1}{2}e^{-2x}\,dx \\ &= -\frac{1}{2}\left(x^2 + x + \frac{1}{2}\right)e^{-2x} + C\end{aligned}$$

[5]

Please deduct a mark for each missing constant of integration. **[Total: 10]**