Maths for Actuarial Science Coursework 1: Answers

1. (i) General term is
$$ {}_{18}C_r (x^3)^{18-r} \left(\frac{3}{x^2}\right)^r $$
We need $54 - 5r = 14$, i.e. $r = 8$. **[2]**
Coefficient of x^{14} is
$$ \frac{18!}{8!10!} 3^8 = 2.3^{10}.11.13.17 $$
[2]
(ii) $(1+x)^6 = 1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$. **[1]**
Therefore
$$ \begin{aligned} (1+(1+z)z)^6 &= 1 + 6z(1+z) + 15z^2(1+z)^2 + 20z^3(1+z)^3 + 15z^4(1+z)^4 + \cdots \\ &= 1 + 6z + 21z^2 + 50z^3 + 90z^4 + \cdots. \end{aligned} $$
[2]
[Total: 7]

2. (i) Circles meet when
$$ x^2 + y^2 - 2x - 4y - 4 = x^2 + y^2 - 6x - 2y - 8 $$
i.e. when $4x - 2y = 4$, or $y = 2x + 2$. **[2]**
For points, substitute into equation and get
$$ \left(\frac{1+\sqrt{41}}{5}, \frac{12+2\sqrt{41}}{5}\right) \quad \text{and} \quad \left(\frac{1-\sqrt{41}}{5}, \frac{12-2\sqrt{41}}{5}\right). $$
[3]
(ii) Centre of ellipse is midway between foci, so centre is $(3,6)$. **[1]**
Major axis of length $2a = 6$, so $a = 3$. **[1]**
Distance from foci to centre is $ae = 1$, so $e = \frac{1}{3}$. **[1]**
Therefore $b^2 = a^2(1-e^2) = 8$. **[1]**
The equation is thus
$$ \frac{(y-6)^2}{9} + \frac{(x-2)^2}{8} = 1. $$
[1]
[Total: 10]

3. (i) Verification of identity. **[5]**
(ii) We have
$$ \sin 5\theta + \sin\theta = 2\sin 3\theta \cos 2\theta = \sin 3\theta $$
and so we must solve
$$ \sin 3\theta (2\cos 2\theta - 1) = 0. $$
[2]

For $\sin 3\theta = 0$ we have general solution

$$\theta = \frac{n\pi}{3}$$

with $n \in \mathbb{Z}$, and for $\cos 2\theta = \frac{1}{2}$ we have

$$\theta = \left(n \pm \frac{1}{6}\right)\pi$$

with $n \in \mathbb{Z}$. (Please deduct a mark for approximate answers, and for not stating the domain of n.) **[3]**

[Total: 10]

4. (i)(a)

$$3(1+3x)(1+2\ln(1+3x)).$$

[2]

(b)

$$6x^2\tan(x^3+1)\sec^2(x^3+1).$$

[2]

(ii) $\frac{\mathrm{d}x}{\mathrm{d}t} = 2t+1$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = 1/t$ so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{t(2t+1)}.$$

[2]

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{t(2t+1)}\right)\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{-(4t+1)}{t^2(2t+1)^3}.$$

[2]

[Total: 8]

5. (a)

$$\int \frac{x+2}{1-4x^2}\,dx = \int \frac{5}{4(1-2x)} + \frac{3}{4(1+2x)}\,dx = -\frac{5}{8}\ln(1-2x) + \frac{3}{8}\ln(1+2x) + C.$$

Please deduct a mark for a missing constant of integration. **[4]**

(b) Let $u = \sqrt{2x+1}$, so $x = \frac{u^2-1}{2}$ and $\frac{\mathrm{d}x}{\mathrm{d}u} = u$.

$$\int_0^1 \frac{2x}{\sqrt{2x+1}}\,dx = \int_1^{\sqrt{3}} \frac{u^2-1}{u}u\,du = \left[\frac{u^3}{3} - u\right]_1^{\sqrt{3}} = \frac{2}{3}.$$

[4]

[Total: 8]

6. Let $y = \tan^{-1}x$, i.e. $x = \tan y$ with $-\frac{\pi}{2} < y < \frac{\pi}{2}$. Now

$$\sin 2y = 2\sin y\cos y = 2\tan y\cos^2 y = \frac{2\tan y}{1+\tan^2 y} = \frac{2x}{1+x^2}.$$

[Total: 7]