Maths for Actuarial Science Coursewor 1: Answers

1. Easiest way is to sketch curves and inspect. Algebraically, any method is OK provided all cases are considered. Please deduct marks for each time a case is left out. One way: Need to solve
$$\frac{2x+3}{x-5} < \frac{4x+12}{x+1} \tag{1}$$
when the RHS is positive (which is when $x < -3$ or $x > -1$) and
$$\frac{2x+3}{x-5} < \frac{-4x-12}{x+1} \tag{2}$$
otherwise. For eqn (1) rearrange and solve to find that $x > \frac{13+\sqrt{673}}{4}$ or $x < \frac{13-\sqrt{673}}{4}$ or $-1 < x < 5$. From the other case we obtain $\frac{3-\sqrt{1377}}{12} < x < -1$. (The last fraction also simplifies to $\frac{1-3\sqrt{17}}{4}$ but this is not required.) Thus the answer is
$$x \in \Big(-\infty, \frac{13-\sqrt{673}}{4}\Big) \cup \Big(\frac{3-\sqrt{1377}}{12}, -1\Big) \cup \Big(-1, 5\Big) \cup \Big(\frac{13+\sqrt{673}}{4}, \infty\Big)$$
[Total: 8]

2. The distance from focus to centre is $ae = 2$. **[1]**
As the major axis has length $2a = 10$ this implies that $a = 5$ and $e = 2/\sqrt{25}$. So $b^2 = 25(1 - 4/25) = 21$. **[2]**
Therefore we have
$$\frac{(y-6)^2}{25} + \frac{(x-4)^2}{21} = 1.$$
[3]
[Total: 6]

3. First note that $2\cos 2\theta + 4\sin\theta\cos\theta = 2\cos 2\theta + 2\sin 2\theta$. **[1]**
Then $2\cos 2\theta + 2\sin 2\theta = R\cos(2\theta - \alpha)$ where
$$R^2 = 8 \quad \text{so } R = 2\sqrt{2} \quad \text{and} \quad \alpha = \tan^{-1}(2/2) = \frac{\pi}{4}.$$
[3]
So must solve
$$2\sqrt{2}\cos(2\theta - \frac{\pi}{4}) = \sqrt{2}$$
i.e.
$$\cos(2\theta - \frac{\pi}{4}) = \frac{1}{2}$$
in the range $0 \le \theta \le 2\pi$.
General solution:
$$2\theta - \frac{\pi}{4} = \pm\frac{\pi}{3} + 2n\pi$$
[3]
Therefore $2\theta = \frac{7\pi}{12}, \frac{23\pi}{12}, \frac{31\pi}{12}, \frac{47\pi}{12}$ and hence $\theta = \frac{7\pi}{24}, \frac{23\pi}{24}, \frac{31\pi}{24}, \frac{47\pi}{24}$.
(Please deduct a mark for approximate (ie decimal) answers.) **[3]**
[Total: 10]

4. (i)(a)

$$\frac{\sec x\tan x+\sec^2 x}{\sec x+\tan x}=\sec x.$$

[2]

(b)

$$3(\sin^2 x\cos x\cos 3x-\sin^3 x\sin 3x-\cos^2 x\sin x\sin 3x+\cos^3 x\cos 3x)=3\cos 4x.$$

[2]

(ii) $\frac{\mathrm{d}x}{\mathrm{d}t}=8t-5$ and $\frac{\mathrm{d}y}{\mathrm{d}t}=3t^2-2t+1$. Hence $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}t}(\frac{\mathrm{d}x}{\mathrm{d}t})^{-1}=\frac{3t^2-2t+1}{8t-5}$. **[2]**

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{3t^2-2t+1}{8t-5}\right)\frac{\mathrm{d}t}{\mathrm{d}x}=\frac{24t^2-30t+2}{(8t-5)^3}.$$

[2]
[Total: 8]

5. The fraction simplifies to

$$1-\frac{x^2-9x-9}{(x^2-9)(x+1)}.$$

[2]

Let

$$\frac{x^2-9x-9}{(x^2-9)(x+1)}=\frac{A}{x-3}+\frac{B}{x+3}+\frac{C}{x+1}.$$

[1]

Solve to find $A=-\frac{9}{8}$, $B=\frac{9}{4}$, $C=-\frac{1}{8}$, and hence

$$\frac{x^3}{(x^2-9)(x+1)}=1+\frac{1}{8}\left(\frac{9}{x-3}+\frac{1}{x+1}-\frac{18}{x+3}\right).$$

[3]

Therefore the integral equals

$$x+\frac{1}{8}\left(9\ln(x-3)+\ln(x+1)-18\ln(x+3)\right)+C.$$

[2]
[Total: 8]

6. (a) Let $u=e^x$, and later $\sin\theta=u$. Then

$$\int e^x\sqrt{1-e^{2x}}\,dx=\int\sqrt{1-u^2}\,du=\int\cos^2\theta\,d\theta=\int\frac{1}{2}(1+\cos 2\theta)\,d\theta$$
$$=\frac{1}{2}(\theta+\frac{1}{2}\sin 2\theta)+C=\frac{1}{2}(\sin^{-1}(e^x)+\frac{1}{2}\sin(2\sin^{-1}(e^x)))+C.$$

[5]

(b)

$$\begin{aligned}\int x^2 e^{-2x}\,dx &= -\frac{1}{2}x^2e^{-2x} + \int xe^{-2x}\,dx \\ &= -\frac{1}{2}x^2e^{-2x} - \frac{1}{2}xe^{-2x} + \int \frac{1}{2}e^{-2x}\,dx \\ &= -\frac{1}{2}\left(x^2 + x + \frac{1}{2}\right)e^{-2x} + C\end{aligned}$$

[5]

Please deduct a mark for each missing constant of integration. **[Total: 10]**