Mathematics for Actuarial Science (AS1051)

Coursework 1 (Solutions)

This is an assessed coursework, and will count towards your final grade. Solutions should be handed in to the SEMS general office (C109) by the stated deadline. Late submissions will be penalised.

DEADLINE: Thursday 11/11/2010 at 15:00

1) *i)* The general term in the expansion is [2 marks]

$$\left(2x^4 + \frac{5}{x^3}\right)^{11} \sim \binom{11}{k} \left(2x^4\right)^{11-k} \left(\frac{5}{x^3}\right)^k = \binom{11}{k} 2^{11-k} 5^k x^{4(11-k)-3k}$$

For this to be proportional to x^2 we require $4(11-k) - 3k = 2$ and therefore $k = 6$. Then the coefficient becomes [2 marks]

$$\binom{11}{6} 2^{11-6} 5^6 = \frac{11!}{6!(11-6)!} 5^5 5^6 = 2^6 \times 3 \times 5^6 \times 7 \times 11.$$

ii) First we compute [1 mark]

$$(1+x)^6 = x^6 + 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x + 1.$$

Next we replace x by $y^2 + y^3$ [2 marks]

$$\begin{aligned}\left(1 + y^2 + y^3\right)^6 &= \left(y^3 + y^2\right)^6 + 6\left(y^3 + y^2\right)^5 + 15\left(y^3 + y^2\right)^4 + 20\left(y^3 + y^2\right)^3 \\ &\quad + 15\left(y^3 + y^2\right)^2 + 6\left(y^3 + y^2\right) + 1 \\ &= 1 + 6y^2 + 6y^3 + 15y^4 + 30y^5 + 35y^6 + 60y^7 + \mathcal{O}(y^8).\end{aligned}$$

$\boxed{\sum = 7}$

2) *i)* Verification of the identity. [5 marks]

ii) We note that

$$\cos(6\theta) = \cos^2(3\theta) - \sin^2(3\theta) \qquad \text{and} \qquad 4\cos^2\left(\frac{3}{2}\theta\right)\sin^2\left(\frac{3}{2}\theta\right) = \sin^2(3\theta).$$

Therefore we have to solve [2 marks]

$$\cos^2(3\theta) + \cos(3\theta) = 0,$$

which means

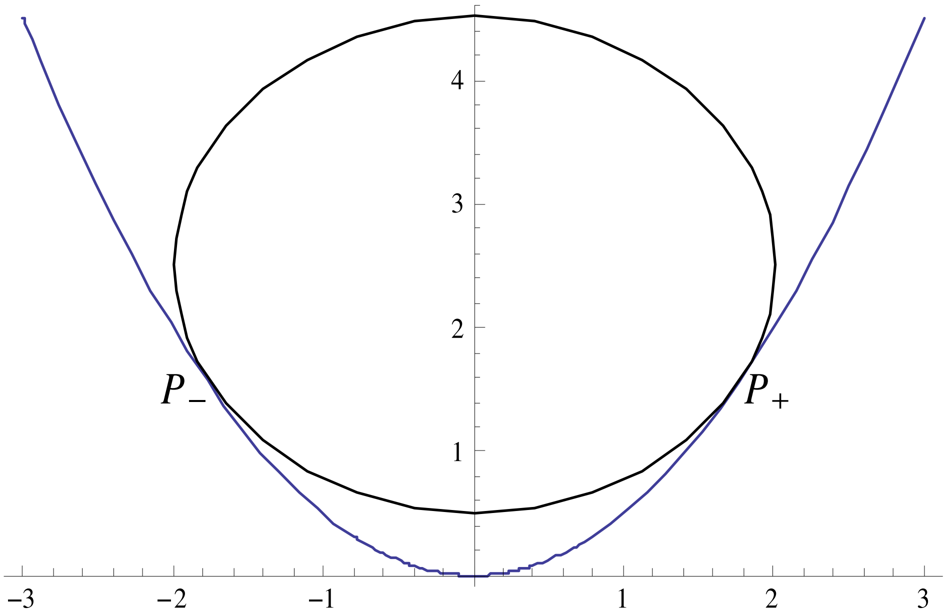
$$\cos(3\theta) = 0 \quad \Rightarrow \ 3\theta = \pm\frac{\pi}{2} + 2\pi n \ \text{ or } \quad \cos(3\theta) = -1 \quad \Rightarrow \ 3\theta = \pm\pi + 2\pi n, \ n \in \mathbb{Z}.$$

Hence, the general solution is [3 marks]

$$\theta = \pm\frac{\pi}{6} + \frac{2}{3}\pi n \qquad \text{or} \qquad \theta = \pm\frac{\pi}{3} + \frac{2}{3}\pi n, \ n \in \mathbb{Z}.$$

$\boxed{\sum = 10}$

3) (i)



[1 mark]

(ii) The equation of the parabola is

$$y = \frac{1}{2}x^2$$

and the equation of the circle is

$$x^2 + (y-a)^2 = 4.$$

Differentiating both equations gives

$$\frac{dy}{dx} = x \qquad \text{and} \qquad 2x + 2(y-a)\frac{dy}{dx} = 0.$$

Since the tangents are the same

$$\Rightarrow 1 + (y-a) = 0 \qquad \Rightarrow (y-a) = -1 \qquad \Rightarrow x^2 + 1 = 4 \qquad \Rightarrow x = \pm\sqrt{3}, y = \frac{3}{2}$$

The points of intersection are $\boxed{P_\pm = (\pm\sqrt{3}, 3/2)}$. [7 marks]

The center results from $(3/2 - a) = -1$, i.e. $\boxed{(0, 5/2)}$.

The intersection with the y-axis is obtained from $(y-5/2)^2 = 4$, i.e. $\boxed{y = 1/2, 9/2}$.

[2 marks]

$\boxed{\sum = 10}$

4) *i)* First we factorise the numerator and denominator

$$f(x) = \frac{x^2(x+1)(x-1)}{(x-1)(x-2)(x^2-2x+4)} = \frac{x^2(x+1)}{(x-2)(x^2-2x+4)}.$$

Since the right hand side is not a proper fraction we use polynomial devision to obtain [2 marks]

$$f(x) = 1 + \frac{5x^2 - 8x + 8}{(x-2)(x^2-2x+4)}.$$

Writing

$$\frac{5x^2 - 8x + 8}{(x-2)(x^2-2x+4)} = \frac{A}{x-2} + \frac{Bx + C}{x^2 - 2x + 4}$$

we find $A = 3$, $B = 2$ and $C = 2$, such that [4 marks]

$$f(x) = 1 + \frac{3}{x-2} + \frac{2(x+1)}{x^2 - 2x + 4}.$$

ii) We compute

$$\begin{aligned}\frac{df(x)}{dx} &= \frac{d}{dx}\left(\frac{3}{x-2}\right) + \frac{d}{dx}\left(\frac{2(x+1)}{x^2-2x+4}\right) \\ &= -\frac{3}{(x-2)^2} + \frac{2}{x^2-2x+4} - \frac{2(x+1)(2x-2)}{(x^2-2x+4)^2}\end{aligned}$$

At $x = 0$ this becomes [4 marks]

$$\frac{df(0)}{dx} = -\frac{3}{4} + \frac{2}{4} - \frac{-4}{4^2} = 0.$$

[1 mark]

iii) We compute

$$\begin{aligned}\int f(x)dx &= \int 1dx + \int \frac{3}{x-2}dx + \int \frac{2(x+1)}{x^2-2x+4}dx \\ &= x + 3\ln(x-2) + 2\int \frac{x-2}{x^2-2x+4}dx + 2\int \frac{2}{x^2-2x+4}dx \\ &= x + 3\ln(x-2) + \ln(x^2-2x+4) + 4\int \frac{1}{x^2-2x+4}dx.\end{aligned}$$

We evaluate [3 marks]

$$\begin{aligned}\int \frac{1}{x^2-2x+4}dx &= \int \frac{1}{(x-1)^2+3}dx \\ &= \int \frac{1}{y^2+3}dx && \text{with } y=x+1 \\ &= \frac{1}{\sqrt{3}}\int \frac{1}{z^2+1}dx && \text{with } z=\frac{y}{\sqrt{3}} \\ &= \frac{1}{\sqrt{3}}\arctan\left[\frac{1}{\sqrt{3}}(x-1)\right].\end{aligned}$$

Therefore [2 marks]

$$\int f(x)dx = x + \ln\left[(x-2)^3(x^2-2x+4)\right] + \frac{4}{\sqrt{3}}\arctan\left[\frac{1}{\sqrt{3}}(x-1)\right].$$

$\boxed{\sum = 16}$

5) For

$$x(t) = t^3+3t+1 \qquad \text{and} \qquad y(t) = \ln(3t^2-3)$$

we compute

$$\frac{dx}{dt} = 3t^2+3 \qquad \text{and} \qquad \frac{dy}{dt} = \frac{2t}{t^2-1}$$

and subsequently [2 marks]

$$\frac{d^2x}{dt^2} = 6t \qquad \text{and} \qquad \frac{d^2y}{dt^2} = -\frac{2(t^2+1)}{(t^2-1)^2}.$$

Therefore [2 marks]

$$\frac{dy}{dx} = \frac{2t}{3(t^4-1)}, \qquad \frac{d^2y}{dt^2}\frac{d^2t}{dx^2} = -\frac{1+t^2}{3t(t^2-1)^2}$$

and

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = \frac{d}{dt}\left(\frac{2t}{3(t^4-1)}\right)\frac{1}{3t^2+3} \\ &= -\frac{2\left(3t^4+1\right)}{9\left(t^2-1\right)^2\left(t^2+1\right)^3},\end{aligned}$$

such that [3 marks]

$$p(t) = \frac{2\left(3t^5+t\right)}{3\left(t^2+1\right)^4}.$$

$\boxed{\sum = 7}$