Mathematics for Actuarial Science (AS1051)

Coursework 1 (Solutions)

This is an assessed coursework, and will count towards your final grade. Solutions should be handed in to the SEMS general office (C108) by the stated deadline. Late submissions will be penalised.

DEADLINE: Thursday 10/11/2011 at 15:00

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1) *i)* The solution is $w = -49$, $x = 5$, $y = 19$ and $z = 6$. [5 marks]

ii) We use the identity $\log_a c = \log_b c / \log_b a$ to re-write [5 marks]

$$\log_3(x) = \log_9(|x-12|) = \frac{\log_3(|x-12|)}{\log_3 9}.$$

Next we use $\log_3 9 = \log_3 3^2 = 2\log_3 3 = 2$ and $2\log_3(x) = \log_3(x^2)$, such that

$$\log_3(x^2) = \log_3(|x-12|) \quad \Rightarrow \quad x^2 = |x-12|\,.$$

Thus

$$\begin{aligned} &\text{for } x < 12: \quad x^2 + x - 12 = 0 \quad \Rightarrow \quad x_1 = 3,\ x_2 = -4, \\ &\text{for } x > 12: \quad x^2 - x + 12 = 0 \quad \Rightarrow \quad \text{no real solution.} \end{aligned}$$

Since $\log_3(x)$ is only defined for $x > 0$ we have to dicard the solution x_2 and the only viable solution is $\underline{x = 3}$. $\boxed{\sum = 10}$

2) *i)* We have [4 marks]

$$\begin{aligned} \cos(2\theta - 2\phi) &= \cos(2\theta)\cos(2\phi) + \sin(2\theta)\sin(2\phi) \\ &= 4\cos\theta\cos\phi\sin\theta\sin\phi + \left[\cos^2\theta - \sin^2\theta\right]\left[\cos^2\phi - \sin^2\phi\right] \\ &= \cos^2\theta\cos^2\phi - \sin^2\theta\cos^2\phi + 4\cos\theta\sin\theta\sin\phi\cos\phi \\ &\quad -\cos^2\theta\sin^2\phi + \sin^2\theta\sin^2\phi \end{aligned}$$

We also have

$$\begin{aligned} &[\cos(\theta-\phi) + \sin(\theta-\phi)]\,[(\cos\theta - \sin\theta)\cos\phi + \cos\theta\sin\phi + \sin\theta\sin\phi] \\ = &[\cos\theta\cos\phi + \sin\theta\cos\phi - \cos\theta\sin\phi + \sin\theta\sin\phi] \\ &\times[\cos\theta\cos\phi - \sin\theta\cos\phi + \cos\theta\sin\phi + \sin\theta\sin\phi] \\ = &\cos^2\theta\cos^2\phi - \sin^2\theta\cos^2\phi + 4\cos\theta\sin\theta\sin\phi\cos\phi \\ &-\cos^2\theta\sin^2\phi + \sin^2\theta\sin^2\phi, \end{aligned}$$

which proves the identity.

ii) Re-write [6 marks]

$$2\cos 2\theta + 4\sin\theta\cos\theta = 2\cos 2\theta + 2\sin 2\theta = R\cos(2\theta - \alpha),$$

and determine R and α. Then

$$R^2 = 2^2 + 2^2 = 8 \quad \text{and} \quad \alpha = \arctan\frac{2}{2} = \frac{\pi}{4},$$

such that

$$2\sqrt{2}\cos(2\theta - \frac{\pi}{4}) = \sqrt{2} \quad \Rightarrow \quad \cos(2\theta - \frac{\pi}{4}) = \frac{1}{2} \text{ for } 2\pi \leq \theta \leq 4\pi.$$

The general solution is

$$2\theta - \frac{\pi}{4} = \pm\frac{\pi}{3} + 2n\pi \quad \Rightarrow \theta = \frac{7\pi}{24} + n\pi \quad \text{and} \quad \theta = -\frac{\pi}{24} + n\pi \quad \text{with} \quad n \in \mathbb{Z}.$$

In the required range we find:

$$\theta = \frac{54\pi}{24}, \frac{79\pi}{24}, \frac{71\pi}{24}, \frac{95\pi}{24}.$$

$\boxed{\sum = 10}$

3) *i*) [3 marks]

$$\mathcal{L}_1 : \quad 3 = \frac{y-1}{x-1} \quad \Rightarrow \quad y = 3x - 2$$

$$\mathcal{L}_2 : \quad m = \frac{0+9}{50-5} \quad \Rightarrow \quad m = \frac{1}{5} = \frac{y+9}{x-5} \quad \Rightarrow \quad y = \frac{x}{5} - 8$$

ii) $\mathcal{L}_2^\perp$ has gradient $m = -5$ and passes $(0, -10)$, therefore [4 marks]

$$\mathcal{L}_2^\perp : \quad -5 = \frac{y+10}{x-0} \quad \Rightarrow \quad y = -5x - 10$$

For rhe point of intersection P between $\mathcal{L}_1$ and $\mathcal{L}_2^\perp$ we require

$$-5x - 10 = \frac{x}{5} - 8 \Rightarrow \quad x = -1 \Rightarrow \quad y = -5, \text{ i.e.} \quad \underline{P = (-1, -5)}.$$

$\boxed{\sum = 7}$

4) *i*) We have [10 marks]

$$\frac{2x^5 + x + 4}{(x-1)\,(x^2 - 3x + 2)\,(x^2 + x - 2)} = \frac{2x^5 + x + 4}{(x+2)(x-2)\,(x-1)^3}$$

This is not a proper fraction, so we write

$$\begin{aligned}
\frac{2x^5 + x + 4}{(x+2)(x-2)\,(x-1)^3} &= 2 + \frac{2x^5 + x + 4 - 2(x+2)(x-2)\,(x-1)^3}{(x+2)(x-2)\,(x-1)^3} \\
&= 2 + \frac{6x^4 + 2x^3 - 22x^2 + 25x - 4}{(x-2)(x-1)^3(x+2)} \\
&= 2 + \frac{A}{(x-2)} + \frac{B}{(x+2)} + \frac{C}{(x-1)} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3}
\end{aligned}$$

Therefore

$$\begin{aligned}6x^4+2x^3-22x^2+25x-4 &= Ax^4+Bx^4+Cx^4-Ax^3-5Bx^3-2Cx^3+Dx^3\\ &\quad -3Ax^2+9Bx^2-3Cx^2-Dx^2+Ex^2+5Ax\\ &\quad -7Bx+8Cx-4Dx-2A+2B-4C+4D-4E\end{aligned}$$

Comparing coefficients gives

$$\begin{aligned}-6+A+B+C &= 0,\\ -2-A-5B-2C+D &= 0,\\ 22-3A+9B-3C-D+E &= 0,\\ -25+5A-7B+8C-4D &= 0,\\ 4-2A+2B-4C+4D-4E &= 0,\end{aligned}$$

and therefore $A=35/2$, $B=-31/54$, $C=-295/27$, $D=-47/9$, $E=-7/3$. Hence

$$f(x)=2+\frac{35}{2(x-2)}-\frac{31}{54(x+2)}+\frac{295}{27(x-1)}-\frac{47}{9(x-1)^2}-\frac{7}{3(x-1)^3}$$

ii) Differentiating $f(x)$ [3 marks]

$$f'(x)=-\frac{35}{2(x-2)^2}+\frac{295}{27(x-1)^2}+\frac{31}{54(x+2)^2}+\frac{94}{9(x-1)^3}+\frac{7}{(x-1)^4},$$

such that $f'(0)=13/4$ and $f'(-1)=71/144$.

iii) Integrating $f(x)$ yields [3 marks]

$$\begin{aligned}\int f(x)dx &= \frac{1}{54}\left[108x+945\log(x-2)-590\log(x-1)-31\log(x+2)\right.\\ &\quad \left.+\frac{282}{x-1}+\frac{63}{(x-1)^2}\right]\end{aligned}$$

$\boxed{\sum = 16}$

5) We use [7 marks]

$$(1+a)^n=\sum_{k=0}^{n}\binom{n}{k}a^k$$

on both sides of the equation

$$(1+x)^n(1-x)^n=(1-x^2)^n$$

$$\begin{aligned}\left[\sum_{t=0}^{n}\binom{n}{t}x^t\right]\left[\sum_{l=0}^{n}\binom{n}{l}(-x)^l\right] &= \sum_{s=0}^{n}\binom{n}{s}(-x^2)^s\\ \sum_{t=0}^{n}\sum_{l=0}^{n}\binom{n}{t}\binom{n}{l}(-1)^l x^{t+l} &= \sum_{s=0}^{n}\binom{n}{s}(-x^2)^s\end{aligned}$$

Comparing coefficients of x^{2k} for some k. On the left hand side for $t < 2k$ when $t + l = 2k$

$$\sum_{t=0}^{2k} (-1)^{2k-t} \binom{n}{t} \binom{n}{2k-t}$$

and on the right hand side

$$\binom{n}{k} (-1)^k.$$

Therefore

$$\sum_{t=0}^{2k} (-1)^{k-t} \binom{n}{t} \binom{n}{2k-t} = \binom{n}{k}.$$

$\boxed{\sum = 7}$