Maths for Actuarial Science Answers, 2008

Paper 2 Section A

Question 1:

$$4x^2 - 8x - (y^2 - 6y) = 1 \quad \text{rearranges to} \quad \frac{(y-3)^2}{4} - \frac{(x-1)^2}{1} = 1.$$

This is the standard form for a hyperbola (but with the usual roles of x and y reversed). **[3]**
We see that the centre is at $(1, 3)$ and in the standard notation $b = 1$ and $a = 2$ where $b^2 = a^2(e^2 - 1)$. Therefore $e = \frac{\sqrt{5}}{2}$. Then the foci are at $(1, 3 \pm ae) = (1, 3 \pm \sqrt{5})$ and the asymptotes are given by

$$x - 1 = \pm\frac{b}{a}(y - 3) = \pm\frac{1}{2}(y - 3).$$

This rearranges to $y = 2x + 1$ and $y = -2x + 5$. **[5]**

Question 2: We want to prove that

$$\sum_{i=1}^{n}(2i - 1) = n^2.$$

We proceed by induction. When $n = 1$ we have $1 = 1^2$ which is true. **[2]**
Now assume the result is true for $n = k$; we want that this implies the result for $n = k + 1$. We have

$$\sum_{i=1}^{k+1}(2i - 1) = \sum_{i=1}^{k}(2i - 1) + (2(k + 1) - 1) = k^2 + 2k + 1 = (k + 1)^2.$$

where the second equality follows from the inductive hypothesis. The result now follows by induction. **[6]**

Question 3:

(a) First note that this difference equation is equivalent to, for $n \geq 0$,

$$u_{n+1} - 3u_n = 1\,.$$

We find the general solution of the homogeneous equation. From the lectures, this is $C3^n$ for some constant C. Now, we find a particular solution to the complete equation. Given the right hand side, we look for it in the form $u_n = an + b$ for some constants a, b to be determined. Inserting we get,

$$-2an + a - 2b = 1\,,$$

so

$$\begin{cases} -2a = 0\,, \\ a - 2b = 1 \end{cases} \quad \Leftrightarrow \quad \begin{cases} a = 0\,, \\ b = -\frac{1}{2} \end{cases}$$

Collecting everything, we obtain the general solution as $u_n = C3^n - \frac{1}{2}$. **[3]**

(b) Again, here we look for the general solution of the homogeneous equation first. This involves the auxiliary equation

$$a^2 - 3a - 4 = 0\,,$$

with roots 4 and -1.So, the general solution of the homogeneous equation reads

$$u_n = A4^n + B(-1)^n\,.$$

To this, we need to add a particular solution to the complete equation. Given the form of the inhomogeneous term, we simply try $u_n = a$ and get $a - 3a - 4a = 1$ that is $a = -\frac{1}{6}$. We can now impose the initial conditions on the complete general solution $u_n = A4^n + B(-1)^n - \frac{1}{6}$ to fix A and B:

$$A + B - \frac{1}{6} = \frac{2}{3} \quad \text{and} \quad 4A - B - \frac{1}{6} = \frac{2}{3}\,,$$

which yields

$$A = \frac{1}{3} \quad \text{and} \quad B = \frac{1}{2}\,.$$

Finally, for all $n \geq 0$,

$$u_n = \frac{4^n}{3} + \frac{(-1)^n}{2} - \frac{1}{6}$$

[5]

Question 4:

(a) We compute the determinant of M expanding over the first row:

$$|M| = \begin{vmatrix} 1 & 1 & 0 \\ 3 & a & 2 \\ 2 & 1 & a \end{vmatrix} = \begin{vmatrix} a & 2 \\ 1 & a \end{vmatrix} - \begin{vmatrix} 3 & 2 \\ 2 & a \end{vmatrix} = a^2 - 3a + 2\,.$$

It is zero for $a = 1$ or $a = 2$, in which cases M is not invertible. **[3]**

(b) Any method will do. We use for instance the comatrix formula

$$M^{-1} = \frac{1}{|M|} A^T ,$$

where A^T is the transpose of the comatrix. We compute the nine minors

$$M_{11} = \begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} = -2 \ , \quad M_{12} = \begin{vmatrix} 3 & 2 \\ 2 & 0 \end{vmatrix} = -4 \ , \quad M_{13} = \begin{vmatrix} 3 & 0 \\ 2 & 1 \end{vmatrix} = 3 ,$$

$$M_{21} = \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} = 0 \ , \quad M_{22} = \begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix} = 0 \ , \quad M_{23} = \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -1 ,$$

$$M_{31} = \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = 2 \ , \quad M_{32} = \begin{vmatrix} 1 & 0 \\ 3 & 2 \end{vmatrix} = 2 \ , \quad M_{33} = \begin{vmatrix} 1 & 1 \\ 3 & 0 \end{vmatrix} = -3 ,$$

and then the comatrix (don't forget the signs in the cofactors),

$$A = \begin{pmatrix} -2 & 4 & 3 \\ 0 & 0 & 1 \\ 2 & -2 & -3 \end{pmatrix} .$$

Finally, recalling that $|M| = 2$ when $a = 0$, we get

$$M^{-1} = \frac{1}{2} \begin{pmatrix} -2 & 0 & -2 \\ 4 & 0 & -2 \\ 3 & 1 & -3 \end{pmatrix} .$$

[5]

Question 5:

We start by writing z in polar form: $z = r(\cos\theta + i\sin\theta)$ for $r > 0$ and $\theta \in [0, 2\pi)$. Then, using De Moivre's theorem we have

$$z^4 = r^4(\cos 4\theta + i \sin 4\theta) .$$

Noting that $\frac{1}{2} + i\frac{\sqrt{3}}{2} = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$, we obtain $r = 1$ and $\theta = \frac{\pi}{24} + n\frac{\pi}{2}$. Keeping only the values in $[0, 2\pi)$, we obtain four solutions $\theta_1 = \frac{\pi}{24}$, $\theta_2 = \frac{13\pi}{24}$, $\theta_3 = \frac{25\pi}{24}$, $\theta_4 = \frac{37\pi}{24}$ giving

$$z_1 = \cos\frac{\pi}{24} + i\sin\frac{\pi}{24} \ , \quad z_2 = \cos\frac{13\pi}{24} + i\sin\frac{13\pi}{24}$$

$$z_3 = \cos\frac{25\pi}{24} + i\sin\frac{25\pi}{24} \ , \quad z_1 = \cos\frac{37\pi}{24} + i\sin\frac{37\pi}{24}$$

[8]

Question 6:

We define the relation ρ on the set of complex numbers $\mathbb{C}$ by

$$z' \,\rho\, z \Leftrightarrow \exists R \in \mathbb{C}\ ,\ R \neq 0 \text{ such that } z' = Rz\,.$$

(a) An equivalence relation on a set S is a relation ρ which is:

- reflexive: $\forall x \in S$, $x \,\rho\, x$,
- symmetric: $\forall x, y \in S$, $x \,\rho\, y \Rightarrow y \,\rho\, x$,
- transitive: $\forall x, y, z \in S$, $x \,\rho\, y$ and $y \,\rho\, z \Rightarrow x \,\rho\, z$

[3]

(b) For the given relation we need to check these three properties.

- reflexive: $\forall z \in \mathbb{C}$, $z = 1.z$ so $R = 1$ is suitable.
- symmetric: Suppose $z' \,\rho\, z$ then there exists $R \neq 0$ such that $z' = Rz$. As $R \neq 0$, it is invertible so $z = R^{-1}z'$ and we deduce $z \,\rho\, z'$,
- transitive: Suppose $z'' \,\rho\, z'$ and $z' \,\rho\, z$ then there exists $R \neq 0$ and $R' \neq 0$ such that $z'' = R'z'$ and $z' = Rz$. Thus, $z'' = R'Rz$ and $R'R \neq 0$ so $z'' \,\rho\, z$.

[5]

Paper 2 Section B

Question 7:

(a) We have

$$\sin 5\theta + \sin\theta = 2\sin 3\theta\cos 2\theta.$$

Therefore we must solve

$$\sin 3\theta(2\cos 2\theta - 1) = 0.$$

[4]

If $\sin 3\theta = 0$ then

$$\theta = \frac{n\pi}{3}$$

with $n \in \mathbb{Z}$. If $\cos 2\theta = 1/2$ then

$$2\theta = \pm\frac{\pi}{3} + 2n\pi \quad \text{so} \quad \theta = \left(n \pm \frac{1}{6}\right)\pi$$

with $n \in \mathbb{Z}$. **[6]**

(b) We have

$$\cos 4x = 2\cos^2 2x - 1.$$

Therefore

$$\begin{aligned}\cos 4x &= 2(2\cos^2 x - 1)^2 - 1 \\ &= 2(1 - 2\sin^2 x)^2 - 1 \\ &= 2(1 + 4\sin^4 x - 4\sin^2 x) - 1 \\ &= 8\sin^4 x - 8\sin^2 x + 1.\end{aligned}$$

[6]

(c) For $-1 \leq x \leq 1$ the value of $\sin^{-1} x$ is defined to be the unique y such that $x = \sin y$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$. **[3]**

(d) Let

$$4\cos x + 3\sin x = R\cos(x - \alpha).$$

Expanding we have

$$4\cos x + 3\sin x = R\cos x\cos\alpha + R\sin x\sin\alpha$$

and comparing coefficients we obtain

$$R\cos\alpha = 4 \quad \text{and} \quad R\sin\alpha = 3.$$

Therefore $R^2 = 16+9 = 25$ so $R = 5$. Thus the maximum value occurs when $\cos(x - \alpha) = -1$, and equals $1/3$. Similarly the minimum value occurs when $\cos(x - \alpha) = +1$, and equals $1/13$. **[7]**

Question 8:

For $n \in \mathbb{N}$, define

$$u_n = \frac{n(n+1)(2n+1)}{6}\,.$$

(a) For all $j \geq 1$,

$$\begin{aligned} u_j - u_{j-1} &= \frac{j(j+1)(2j+1)}{6} - \frac{j(j-1)(2j-1)}{6} \\ &= \frac{j}{6}\left(2j^2 + 3j + 1 - 2j^2 + 3j - 1\right) \\ &= j^2\,. \end{aligned}$$

[7]

(b) This question uses a method seen in the lectures when talking about the "difference test" for series.

$$A_n = \frac{1}{n}\sum_{j=1}^{n}\frac{j^2}{n^2} = \frac{1}{n^3}\sum_{j=1}^{n}(u_j - u_{j-1}) = \frac{1}{n^3}(u_n - u_0)\,.$$

So, for $n \geq 1$,

$$A_n = \frac{n(n+1)(2n+1)}{6n^3}\,.$$

[10]

(c) On the one hand, directly from the previous expression, we get

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty}\frac{n(n+1)(2n+1)}{6n^3} = \frac{2}{6} = \frac{1}{3}\,.$$

On the other hand,

$$\int_0^1 x^2 dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}\,.$$

[9]

Question 9:

(a) Augmnented matrix

$$\begin{pmatrix} 1 & -3 & -2 & 2 \\ -2 & 1 & -1 & 1 \\ -1 & 2 & a^2 & a \end{pmatrix}.$$

After row reduction get

$$\begin{pmatrix} 1 & -3 & -2 & 1 \\ 0 & -5 & -5 & 5 \\ 0 & 0 & a^2-1 & a+1 \end{pmatrix}.$$

$a = 1$: No solutions, $a = 0$: Unique solution, $a = -1$: Infinite number of solutions **[13]**

(b) Augmnented matrix

$$\begin{pmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ -3 & 1 & 2 & 0 & 1 & 0 \\ -2 & -1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

After first set of row reductions and a swap

$$\begin{pmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & -5 & -1 & 3 & 1 & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 \end{pmatrix}.$$

Next set of row operations

$$\begin{pmatrix} 1 & 0 & 0 & 2/5 & 1/5 & -3/5 \\ 0 & -5 & 0 & 4 & 2 & -1 \\ 0 & 0 & -1 & -1 & -1 & 1 \end{pmatrix}.$$

multiply bottom two rows by $-1/5$ and -1 to get

$$\begin{pmatrix} 1 & 0 & 0 & 2/5 & 1/5 & -3/5 \\ 0 & 1 & 0 & -4/5 & -2/5 & 1/5 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{pmatrix}.$$

with the inverse given by the right 3 columns. **[13]**