# Maths for Actuarial Science Answers, 2009

## Paper 1 Section A

**Question 1:**

$$\frac{x^2 - 10x + 15}{(1+x)(2-x)^2} = \frac{A}{1+x} + \frac{B}{2-x} + \frac{C}{(2-x)^2}.$$

Solving we find $A = \frac{26}{9}$, $B = \frac{17}{9}$, and $C = -\frac{1}{3}$. **[5]**
Hence

$$\int \frac{x^2 - 10x + 15}{(1+x)(2-x)^2}\, dx = \frac{26}{9}\ln(1+x) - \frac{17}{9}\ln(2-x) - \frac{1}{3(2-x)} + K.$$

**[3]**

**Question 2:**

The circles have centres $(1, 2)$ and $(3, 1)$, and the line passing through the two centres is $x + 2y = 5$. **[3]**
Circles meet when

$$x^2 + y^2 - 2x - 4y - 4 = x^2 + y^2 - 6x - 2y - 8$$

i.e. when $4x - 2y = 4$, or $y = 2x + 2$. **[2]**
For points, substitute into equation and get

$$\left(\frac{1+\sqrt{41}}{5}, \frac{12+2\sqrt{41}}{5}\right) \quad \text{and} \quad \left(\frac{1-\sqrt{41}}{5}, \frac{12-2\sqrt{41}}{5}\right).$$

**[3]**

**Question 3:**

(a) Let $u = e^x$. Then

$$\int \frac{e^x}{1-e^{2x}}\, dx = \int \frac{1}{1-u^2}\, du = \int \frac{1}{2}\left(\frac{1}{1-u} + \frac{1}{1+u}\right) du = \frac{1}{2}\ln\left(\frac{1+e^x}{1-e^x}\right) + C.$$

**[4]**

(b)

$$\int \frac{5}{2x^2+5}\, dx = \frac{5}{2}\int \frac{1}{x^2 + \frac{5}{2}}\, dx = \sqrt{\frac{5}{2}}\tan^{-1}\left(x\sqrt{\frac{2}{5}}\right) + C.$$

**[4]**

**Question 4:**

Verify the first identity. **[3]**
Differentiate the given equation $n$ times and collect terms to deduce the second identity. **[5]**

**Question 5:**

The integrating factor here is

$$\exp(\int \cot x dx) = \exp(\int \frac{\cos x}{\sin x} dx) = \exp(\ln \sin x) = \sin x\,.$$

**[3]**

So we know that the general solution of this equation reads

$$f(x) = \frac{1}{\sin x}\left(\int \sin x \operatorname{cosec} x\, dx + C\right)\,,$$

where $C$ is a constant.

**[3]**

Finally

$$f(x) = \frac{x + C}{\sin x}\,.$$

It is defined for $\sin x \neq 0$ i.e. for $x \neq n\pi$, $n \in \mathbb{Z}$.

**[2]**

**Question 6:**

1. This is a direct application of the identity

$$(a+b)^2 - (a-b)^2 = 4ab$$

with $a = \frac{x^2}{2}$ and $b = \frac{1}{2x^2}$ defined for $x \neq 0$.

**[2]**

2. We use the formula seen in the lectures

$$L = \int_a^b \sqrt{1 + \left(\frac{df}{dx}\right)^2} dx$$

with $f(x) = \frac{x^3}{6} + \frac{1}{2x}$, $a = 1$ and $b = 2$. So

$$L = \int_1^2 \sqrt{1 + \left(\frac{x^2}{2} - \frac{1}{2x^2}\right)^2} = \int_1^2 \left(\frac{x^2}{2} + \frac{1}{2x^2}\right) dx$$

using the first part.

**[3]**

Now

$$\int_1^2 \left(\frac{x^2}{2} + \frac{1}{2x^2}\right) dx = \left[\frac{x^3}{6} - \frac{1}{2x}\right]_1^2 = \frac{17}{12}\,.$$

**[3]**

## Section B

### Question 7:

(a) The Maclaurin series of a function $f$ up to quadratic term is given by

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2.$$

Differentiating we find that

$$f'(x) = \frac{xe^x}{(1+x)^2} \quad \text{and} \quad f''(x) = \frac{e^x(1+x)^3 - 2(1+x)xe^x}{(1+x)^4}.$$

Thus we obtain

$$f(x) \approx 1 + \frac{x^2}{2}.$$

**[9]**

(b) When $n = 1$ the result is easy. Now suppose the result is true for $n = k$; we need that this implies it is true for $n = k + 1$. Differentiating $f^{(k)}(x)$ we get

$$3^{k-1}(3e^{3x} + 3(3x + n)e^{3x}) = 3^{k-1}(9x + 3n + 3)e^{3x}$$

which simplifies to give the desired expression for $f^{(k+1)}(x)$. Hence the result follows by induction. **[8]**

(c) Integrating $I_n$ by parts we obtain

$$I_n = [x^n e^x]_0^1 - \int_0^1 nx^{n-1}e^x \, dx = e - nI_{n-1}$$

as required.
It is easy to calculate that $I_0 = e - 1$, and then we deduce that $I_1 = 1$, $I_2 = e - 2$, and $I_3 = 6 - 2e$. **[9]**

### Question 8:

Similar exercises have been covered for the sphere and the torus.

1. $f$ satisfies the equation of the ellipse which we write

$$f(x)^2 = \frac{r^2}{2}\left(1 - \frac{(x-r)^2}{r^2}\right),$$

and since it describes the upper-half, we take to positive square root

$$f(x) = \frac{1}{\sqrt{2}}\sqrt{r^2 - (x-r)^2}.$$

Finally, the full upper half ellipse is described when $x$ runs from 0 to $2r$.

**[5]**

2. We use the formula

$$A = \int_a^b 2\pi f(x)\sqrt{1+\left(\frac{df}{dx}\right)^2}\,dx\,,$$

with $f$ given above, $a = 0$ and $b = 2r$. So

$$\begin{aligned} A &= \pi\sqrt{2}\int_0^{2r} \sqrt{r^2-(x-r)^2}\sqrt{1+\left(\frac{-(x-r)}{\sqrt{2}\sqrt{r^2-(x-r)^2}}\right)^2}dx \\ &= \pi\sqrt{2}\int_0^{2r} \sqrt{r^2-(x-r)^2+\frac{(x-r)^2}{2}}dx \\ &= \pi\sqrt{2}\int_0^{2r} \sqrt{r^2-\frac{(x-r)^2}{2}}dx \end{aligned}$$

**[6]**

3. We use the change of variables $x - r = r\sqrt{2}\sin u$ which yields $dx = r\sqrt{2}\cos u du$ and the following bounds. When $x = 0$ $\sin u = -\frac{\sqrt{2}}{2}$ so $u = -\frac{\pi}{4}$. When $x = 2r$, we obtain similarly $u = \frac{\pi}{4}$.

**[4]**

So we can write

$$\begin{aligned} A &= \pi\sqrt{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{r^2(1-\sin^2 u)}\, r\sqrt{2}\cos u\, du \\ &= 2\pi r^2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2 u\, du = 2\pi r^2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1+\cos 2u}{2}\, du \\ &= \pi r^2\left[u+\frac{\sin 2u}{2}\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\,. \end{aligned}$$

Hence

$$A = \pi r^2\left(\frac{\pi}{2}+1\right)\,.$$

**[11]**

**Question 9:**

1. From $h(x) = \frac{1}{f^2(x)}$ we get

$$\frac{dh}{dx} = -2\frac{f'}{f^3}$$

and inserting in

$$\frac{2}{f^3}f' + \frac{1}{f^2} = x - 1\,.$$

it becomes

$$-h' + h = x - 1\,,$$

or

$$h' - h = 1 - x\,.$$

This a linear first order ODE which can be solved using the integrating factor technique. The latter is $e^{-x}$ here so

$$h(x) = e^x \left[\int (1-x)e^{-x}\,dx + C\right]\,,$$

where $C$ is a constant. Integration by parts give

$$\int xe^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx = -e^{-x}(1+x)\,.$$

So

$$h(x) = e^x\left[-e^{-x} + e^{-x}(1+x) + C\right] = x + Ce^x\,,$$

Finally, an expression for $f$ is

$$f(x) = \frac{1}{\sqrt{x + Ce^x}}\,,$$

whenever this makes sense.

**[13]**

2. First we find the general solution of the homogeneous equation

$$f'' + 8f' + 25f = 0\,.$$

The auxiliary equation is $r^2 + 8r + 25 = 0$ with complex conjugate roots $-4 \pm 3i$. So we get

$$f(x) = e^{-4x}(A\cos 3x + B\sin 3x)\,.$$

Now we look for a particular solution of the complete equation

$$f'' + 8f' + 25f = 48\cos x - 16\sin x\,,$$

in the form $f(x) = \alpha\cos x + \beta\sin x$. Inserting

$$-\alpha\cos x - \beta\sin x + 8(-\alpha\sin x + \beta\cos x) + 25(\alpha\cos x + \beta\sin x) = 48\cos x - 16\sin x\,.$$

Matching the coefficients give $24\alpha + 8\beta = 48$ and $24\beta - 8\alpha = -16$ with solution $\alpha = 2$ and $\beta = 0$. Collecting everything, the general solution of the complete equation is

$$f(x) = e^{-4x}(A\cos 3x + B\sin 3x) + 2\cos x\,.$$

**[13]**