Maths for Actuarial Science Answers, 2010

Paper 1 Section A

Question 1: We have $\cosh^2 x - \sinh^2 x = 1$, and so must solve

$$\cosh^2 x + 9 = 6\cosh x.$$

Rearranging this becomes

$$(\cosh x - 3)^2 = 0$$

and so $\cosh x = 3$. **[5]**
We have

$$\cosh^{-1}(u) = \ln(u + \sqrt{u^2 - 1})$$

and so $\cosh^{-1}(3) = \ln(3 + \sqrt{8})$. As $\cosh(x)$ is an even function, the set of solutions is

$$x = \pm\ln(3 + \sqrt{8}).$$

[3]

Question 2:

(a) The Taylor series of f about c is given by

$$T(f, c) = \sum_{i \geq 0} \frac{f^{(i)}(c)}{i!}(x - c)^i.$$

[2]

(b) Let $f(x) = x^2 \ln x$. Then

$$f'(x) = 2x\ln x + x \quad f''(x) = 2\ln x + 3 \quad f'''(x) = \frac{2}{x}.$$

Therefore

$$f(1) = 0, \quad f'(1) = 1 \quad f''(1) = 3 \quad f'''(1) = 2.$$

Substituting we obtain

$$T(f, c) = (x - 1) + \frac{3}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3.$$

[6]

Question 3:

(a) Let $u = x^2$ and $\frac{\mathrm{d}u}{\mathrm{d}x} = e^{-2x}$. Then

$$\begin{aligned}\int x^2 e^{-2x}\,dx &= -\frac{x^2e^{-2x}}{2} + \int xe^{-2x}\,dx \\ &= -\frac{x^2e^{-2x}}{2} - \frac{xe^{-2x}}{2} + \int \frac{e^{-2x}}{2}\,dx \\ &= -\frac{x^2e^{-2x}}{2} - \frac{xe^{-2x}}{2} - \frac{e^{-2x}}{4} + C.\end{aligned}$$

[4]

(b) Let $u = \sqrt{3x+2}$, so $\frac{\mathrm{d}x}{\mathrm{d}u} = \frac{2u}{3}$. Then

$$\begin{aligned}\int \frac{3x}{\sqrt{3x+2}}\,dx &= \int \frac{u^2-2}{u}\frac{2}{3}u\,du \\ &= \frac{2}{3}\left(\frac{1}{3}u^3 - u\right) + C = \frac{2}{9}(3x-1)\sqrt{3x+2} + C.\end{aligned}$$

[4]

Question 4:

(a) We have $\cosh 0 - e^0\cos 0 = \sin(x) = 0$. So l'Hospital's rule implies that the limit is

$$\lim_{x\to 0}\left(\frac{\sinh x - (e^x\cos x - e^x \sin x)}{3\cos 3x}\right) = -\frac{1}{3}.$$

[4]

(b) We have

$$\lim_{x\to\infty}\left(\frac{5x^2+3x-7}{2-3x^2}\right) = \lim_{x\to\infty}\left(\frac{5+3/x-7/x^2}{2/x^2-3}\right) = -\frac{5}{3}.$$

[4]

Question 5:

$$\begin{aligned}I_n &= \int_0^\pi x^n\cos x\,dx \\ &= [x^n\sin x]_0^\pi - \int_0^\pi nx^{n-1}\sin x dx \\ &= 0 + \left[nx^{n-1}\cos x\right]_0^\pi - \int_0^\pi n(n-1)x^{n-2}\cos x\,dx \\ &= -n\pi^{n-1} - n(n-1)I_{n-2}\end{aligned}$$

[4]

$$\begin{aligned} I_6 &= -6\pi^5 - 30I_4 \\ &= -6\pi^5 - 30(-4\pi^3 - 12I_2) \\ &= -6\pi^5 + 120\pi^3 + 360I_2 \\ &= -6\pi^5 + 120\pi^3 + 360(-2\pi - 2I_0) \\ &= -6\pi^5 + 120\pi^3 - 720\pi \end{aligned}$$

as $I_0 = 0$. **[4]**

Question 6:

Integrating factor is :

$$I = e^{2\int \tan x dx} = e^{-2\ln|\cos x|} = \sec^2 x$$

[4]

Therefore, after multiplying both sides of our equation by I we can write the equation in the form:

$$\frac{d}{dx}(y\sec^2 x) = \sec x \tan x$$

Integrating gives

$$y\sec^2 x = \sec x + c$$

Rearranging yields

$$y = \cos x + c\cos^2 x$$

[4]

Section B

Question 7:

(a) We have

$$\sin(5\theta) = \cos\left(\frac{\pi}{2} - 5\theta\right)$$

and so must solve

$$\cos(7\theta) = \cos\left(\frac{\pi}{2} - 5\theta\right).$$

This has general solution

$$7\theta = 2n\pi \pm \left(\frac{\pi}{2} - 5\theta\right)$$

with $n \in \mathbb{Z}$. Rearranging we obtain

$$\theta = \frac{n\pi}{6} + \frac{\pi}{24} \quad \text{or} \quad \theta = n\pi - \frac{\pi}{4}$$

with $n \in \mathbb{Z}$. **[8]**

(b) $y = \tan^{-1} x$ for $x \in \mathbb{R}$ if and only if $x = \tan y$ with

$$-\frac{\pi}{2} < y < \frac{\pi}{2}.$$

[3]

(c) Let $\alpha = \tan^{-1} a$ and $\beta = \tan^{-1} b$, so $-\frac{\pi}{2} < \alpha, \beta < \frac{\pi}{2}$ and $\tan\alpha = a$ and $\tan\beta = b$. We have

$$\frac{a+b}{1-ab} = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \tan(\alpha+\beta) = \tan(\alpha+\beta+n\pi)$$

(for all $n \in \mathbb{Z}$). Now $\tan^{-1}\left(\frac{a+b}{1-ab}\right)$ must lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, and equal $\alpha + \beta + n\pi$, for some value of n. But by our assumption we have $n = 0$. **[6]**

(d) If the assumption is not satisfied then the righthand side of the equality must be modified to put it in the right range of values (i.e. we do not have $n = 0$ in the above argument). **[2]**

(e) Let $y = \tan^{-1}(x/a)$. Then $x/a = \tan(y)$ and so

$$\frac{1}{a} = \sec^2(y)\frac{dy}{dx}.$$

Therefore

$$\frac{dy}{dx} = \frac{1}{a\sec^2 y} = \frac{1}{a(1+\tan^2 y)} = \frac{1}{a(1+x^2/a^2)} = \frac{a}{(a^2+x^2)}$$

and hence

$$\int \frac{a}{(a^2+x^2)}\,dx = \int \frac{\mathrm{d}y}{\mathrm{d}x}\,dx = y + C.$$

[7]

Question 8:

$f_x = 3x^2 - 4x + 3y$ and $f_y = 3y^2 + 3x - 4y$. $f_{xx} = 6x - 4$, $f_{yy} = 6y - 4$ and $f_{xy} = 3$. **[5]**
At stationary point $f_x = f_y = 0$ and so, subtracting f_y from f_x we obtain:

$$3(x^2 - y^2) - 7(x - y) = 0$$

or

$$3(x - y)(x + y - 7/3) = 0$$

so either $x = y$ or $x + y = 7/3$. **[5]**
The latter is not possible since $y = 7/3 - x$ gives

$$f_x = 3(x - 7/6)^2 + \frac{35}{12}$$

so $f_x > 0$. Therefore we must have $x = y$. This gives $f_x = f_y = 3x^2 - x = x(3x - 1) = 0$ and so $x = 0$ or $x = 1/3$. So stationary points are $(0, 0)$ and $(1/3, 1/3)$. **[6]**
At $(0, 0)$: $f_{xx}f_{yy} - f_{xy}^2 = 7$ i.e $f_{xx}f_{yy} - f_{xy}^2 > 0$ and $fxx = -4 < 0$ and so this is a maximum at which $f = 0$. At $(1/3, 1/3)$: $f_{xx}f_{yy} - f_{xy}^2 = -5$ i.e $f_{xx}f_{yy} - f_{xy}^2 < 0$ and $fxx = -4 < 0$ and so this is a saddle point at which $f = -1/27$. **[4]**

Question 9:

(a) Seek solutions of the form $y\ e^{\lambda x}$. We get the auxiliary equation:

$$\lambda^2 - 2\lambda + 5 = 0$$

So $\lambda_1, \lambda_2 = 1 \pm 2i$. Therefore,

$$y = Ae^{1+2x} + Be^{1-2x}$$

This can be written as

$$y = e^x(C\cos(2x) + D\sin(2x))$$

$(b = 2)$. **[5]**

(b) First seek solutions of form $y\ e^{\lambda x}$. We get the auxiliary equation:

$$\lambda^2 - 2\lambda + 1 = 0.$$

In this case we have repeated roots. Therefore the solution to the homogenous equation is:

$$y_h = Ae^x + Bxe^x$$

[5]

We now seek y_p of the form $y_p = ax^2e^x$: Substituting into the equation we get

$$a(2e^x + 4xe^x + x^2e^x) - 2a(2xe^x + x^2e^x) + a(x^2e^x) = e^x$$

giving $a = 1/2$

$$y_p = \frac{1}{2}x^2e^x$$

therefore

$$\begin{aligned} y &= y_h + y_p \\ &= Ae^x + Bxe^x + \frac{1}{2}x^2e^x \end{aligned}$$

[5]

Now apply boundary condition y(0)=1 gives $1 = A$ To apply the second boundary condition we need: $y' = Ae^x + B(1 + x)e^x + e^x(2x + x^2)$ Now $y'(0) = 0$ so $0 = A + B$ so $B = -1$ Hence

$$y = e^x - xe^x + \frac{1}{2}x^2e^x$$

[5]