Maths for Actuarial Science Answers, 2010

Paper 2 Section A

Question 1:

We have

$$4y^2 - 16y - 9x^2 - 18x - 29 = 4(y-2)^2 - 9(x+1)^2 - 36.$$

Therefore the equation is equivalent to

$$\frac{(y-2)^2}{9} - \frac{(x+1)^2}{4} = 1.$$

[2]

Comparing with the standard form we have $a = 3$ and $b = 2$, and from $b^2 = a^2(e^2 - 1)$ we see that $e = \frac{\sqrt{13}}{3}$. **[2]**

The foci are at

$$(-1, 2 \pm ae) = (-1, 2 \pm \sqrt{13})$$

and the centre is at $(-1, 2)$. **[2]**

The asymptotes are of the form

$$x + 1 = \pm\frac{b}{a}(y-2) = \pm\frac{2}{3}(y-2)$$

which correspond to the equations

$$2y = 3x + 7 \quad \text{and} \quad 2y = 1 - 3x.$$

[2]

Question 2:

Let A be a set with n elements. When $n = 0$ the result is clear as the only subset of A is A itself. **[2]**

Now suppose the result is true when $n = k$. Let A be a set with $k+1$ elements, and choose some element $x \in A$. Let A' be the set obtained from A by removing x. Then every subset of A is determined by its intersection with A', and whether it contains x or not.

There are 2^k subsets of A' by the inductive hypothesis, and each one corresponds to 2 distinct subsets of A (depending on x), and so there are $2 \times 2^k = 2^{k+1}$ subsets of A. The result now follows by induction. **[6]**

Question 3:

(a) Solve first $A_{n+1} = 4A_n$ yields $A_{nh} = A4^n$ Now try a solution of the form $u_n = an + b$. This leads to

$$n(a - 4a) + (a + b - 4b) = n + 2$$

so $a = -1/3$ and $b = -7/9$.
so

$$A_n = A4^n - \frac{1}{3}n - \frac{7}{9}.$$

[4]

(b) Using the auxiliary equation

$$\lambda^2 + 3\lambda - 4 = 0$$

we determine that the solution for the homogeneous equation is

$$A_{nh} = A(1)^n + B(-4)^n.$$

Now try a solution of the form

$$A_n = C5^n$$

we get

$$C5^{n+2} + 3C5^{n+1} - 4C5^n = 5^n.$$

So $C = 1/36$ and

$$A_n = A(1)^n + B(-4)^n + \frac{1}{36}5^n.$$

[4]

Question 4:

(a)

$$|\mathbf{M}| = 10$$

so

$$\mathbf{M}^{-1} = \frac{1}{10}\begin{pmatrix} 6 & -4 \\ -2 & 3 \end{pmatrix}.$$

[4]

(b)

$$\begin{aligned} |\mathbf{M}| &= 1\left|\begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 3 & 0 \end{pmatrix}\right| - 1\left|\begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 0 & 1 & 3 \end{pmatrix}\right| \\ &= -18 + 2(6) - 2 \\ &= -8. \end{aligned}$$

[4]

Question 5:

Write $z = (\cos\theta + i\sin\theta)$. Then

$$\begin{aligned} z^4 &= ((\cos\theta + i\sin\theta))^4 \\ &= (\cos(4\theta) + i\sin(4\theta)) \end{aligned}$$

So then

$$(\cos(4\theta) + i\sin(4\theta)) = (\cos(\pi/2) + i(\sin(\pi/2)).$$

Between 0 and 2π: $\theta = \pi/8, 5\pi/8, 9\pi/8, 13\pi/8$.
Four solutions:

$$z_1 = \cos\left(\frac{\pi}{8}\right) + i\sin\left(\frac{\pi}{8}\right)$$

$$z_2 = \cos\left(\frac{5\pi}{8}\right) + i\sin\left(\frac{5\pi}{8}\right)$$

$$z_3 = \cos\left(\frac{9\pi}{8}\right) + i\sin\left(\frac{9\pi}{8}\right)$$

$$z_4 = \cos\left(\frac{13\pi}{8}\right) + i\sin\left(\frac{13\pi}{8}\right).$$

[**8**]

Question 6:

(a) A relation on a set A is said to be reflexive if $aRa\ \forall a \in A$.
(b) A relation in a set A is said to be symmetric $\forall a, b \in A\ aRb \Rightarrow bRa$. [**4**]

This relation is not reflexive as because, for example, r is not related to r. This relation is symmetric by the definition above. We note that (r, s) and (s, r) also (t, r) and (r, t) are both in R. [**4**]

Section B

Question 7:

(a) We have

$$\begin{aligned}\cos(x) &= \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)\\ &= \left(\cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)\right)\frac{\cos^2(x/2)}{\cos^2(x/2)}\\ &= \frac{1-\tan^2(x/2)}{\sec^2(x/2)}\\ &= \frac{1-t^2}{1+t^2}\end{aligned}$$

as $\sec^2\left(\frac{x}{2}\right) = 1 + \tan^2\left(\frac{x}{2}\right)$. **[7]**

(b) Let $t = \tan(x/2)$. Then

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{2}\sec^2\left(\frac{x}{2}\right) = \frac{1}{2}\left(1+\tan^2\left(\frac{x}{2}\right)\right) = \frac{1}{2}(1+t^2)$$

and hence

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{2}{1+t^2}.$$

[4]

(c) Let $t = \tan(x/2)$. Then

$$\begin{aligned}\int \frac{1}{4\cos x + 5}\,dx &= \int \frac{1}{4\left(\frac{1-t^2}{1+t^2}\right)+5}\frac{2}{1+t^2}\,dt\\ &= \int \frac{2}{4(1-t^2)+5(1+t^2)}\,dt\\ &= \int \frac{2}{t^2+9}\,dt\\ &= \frac{2}{3}\tan^{-1}\left(\frac{t}{3}\right) + C\\ &= \frac{2}{3}\tan^{-1}\left(\frac{1}{3}\tan\left(\frac{x}{2}\right)\right) + C.\end{aligned}$$

[7]

(d) Osborne's rule suggests the identity

$$\cosh(x) = \frac{1+\tanh^2(x/2)}{1-\tanh^2(x/2)}.$$

Now

$$\begin{aligned}
\frac{1+\tanh^2(x/2)}{1-\tanh^2(x/2)} &= \frac{1+\frac{e^x+e^{-x}-2}{e^x+e^{-x}+2}}{1-\frac{e^x+e^{-x}-2}{e^x+e^{-x}+2}} \\
&= \frac{e^x+e^{-x}+2+e^x+e^{-x}-2}{e^x+e^{-x}+2-e^x-e^{-x}+2} \\
&= \frac{e^x+e^{-x}}{2} = \cosh(x).
\end{aligned}$$

[8]

Question 8:

(a) By the ratio test, $a_{n+1}/a_n = \frac{(n+1)(1+\sqrt{5})}{6n}$. This tends to $\frac{1+\sqrt{5}}{6}$ as $n \to \infty$. As $\frac{1+\sqrt{5}}{6} < 1$ the series converges. **[6]**

(b)

$$S_n = 1 + 5x + 9x^2 + \cdots + [1 + 4(n-1)]x^{n-1}$$

Multiply S_n by x to get

$$xS_n = 1x + 5x^2 + 9x^3 + \cdots + [1 + 4(n-1)]x^n$$

Then

$$\begin{aligned}
(1-x)S_n &= 1 + 4x + 4x^2 + \cdots + 4x^{n-1} - [1 + 4(n-1)]x^n \\
&= 1 + 4x(1 + x + \cdots + x^{n-2}) - [1 + 4(n-1)]x^n \\
&= 1 + 4x\frac{1-x^{n-1}}{1-x} - [1 + 4(n-1)]x^n \\
S_n &= \frac{1}{1-x} + 4x\frac{1-x^{n-1}}{(1-x)^2} - \frac{[1 + 4(n-1)]x^n}{1-x}.
\end{aligned}$$

[13]

If $|x| < 1$: $x^n \to 0$ as $n \to \infty$. $nx^n \to 0$ as $n \to \infty$. So

$$\begin{aligned}
\lim_{n\to\infty} S_n &= \frac{1}{1-x} + \frac{4x}{(1-x)^2} \\
&= \frac{1-x+4x}{(1-x)^2} \\
&= \frac{1+3x}{(1-x)^2}.
\end{aligned}$$

[7]

Question 9:

We first form:

$$\begin{pmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 & 1 & 0 \\ 2 & -1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

subtracting three times row 1 from row 2 and subtracting two times of row 1 from row 3 we get

$$\begin{pmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -5 & -2 & -3 & 1 & 0 \\ 0 & -5 & -1 & -2 & 0 & 1 \end{pmatrix}$$

subtracting row 2 from row 3 gives

$$\begin{pmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -5 & -2 & -3 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{pmatrix}$$

now adding 2/5 times row 2 to row 1 gives

$$\begin{pmatrix} 1 & 0 & 1/5 & -1/5 & 2/5 & 0 \\ 0 & -5 & -2 & -3 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{pmatrix}$$

add $(-1/5)$row 3 to row 1 gives and adding 2 times row 3 to row 2 gives

$$\begin{pmatrix} 1 & 0 & 0 & -2/5 & 3/5 & -1/5 \\ 0 & -5 & 0 & -1 & -1 & 2 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{pmatrix}$$

and then

$$\begin{pmatrix} 1 & 0 & 0 & -2/5 & 3/5 & -1/5 \\ 0 & 1 & 0 & 1/5 & 1/5 & -2/5 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{pmatrix}.$$

Therefore

$$\mathbf{M}^{-1} = \begin{pmatrix} -2/5 & 3/5 & -1/5 \\ 1/5 & 1/5 & -2/5 \\ 1 & -1 & 1 \end{pmatrix}.$$

[12]

Form the matrix **A**

$$\begin{pmatrix} 1 & -3 & 4 & 1 \\ -1 & 1 & -7 & 1 \\ 1 & -3 & a^2 & 2a \end{pmatrix}.$$

After Gaussian elimination, we get

$$\begin{pmatrix} 1 & -3 & 4 & 1 \\ 0 & -2 & -3 & 0 \\ 0 & 0 & a^2-4 & 2a-1 \end{pmatrix}.$$

[6]

When a = 1, the system becomes

$$x - 3y + 4z = 1,$$

$$-2y - 3z = 0,$$

$$-3z = 1,$$

and so $x = 23/6$, $y = 1/2$,$z = -1/3$. **[4]**

When a = 0, the system becomes

$$x - 3y + 4z = 1,$$

$$-2y - 3z = 0,$$

$$-4z = -1,$$

and so $x = -9/8$, $y = 3/8$,$z = 1/4$. **[4]**