Maths for Actuarial Science Jan 08: Answers

1. (a) Domain: all $x \neq -\frac{1}{2}$ such that $\frac{x+3}{2x+1} > 0$.
   i.e. $x+3 > 0$ and $2x+1 > 0$ or $x+3 < 0$ and $2x+1 < 0$.
   This gives $x > -\frac{1}{2}$ or $x < -3$. **[3]**
   (b) We need $\frac{x+3}{2x+1} \geq 1$, or $\frac{2-x}{2x+1} \geq 0$.
   Solving as in (a) we see that $-\frac{1}{2} < x \leq 2$. **[5]**
   [Total: 8]

2. We have

$$\tan y = \tan(\frac{\pi}{4} - 2x) = \frac{\tan(\frac{\pi}{4}) - \tan(2x)}{1 + \tan(\frac{\pi}{4})\tan(2x)} = \frac{1 - \frac{2t}{1-t^2}}{1 + \frac{2t}{1-t^2}} = \frac{1 - 2t - t^2}{1 + 2t - t^2}$$

   where $t = \tan x$. **[4]**
   Now $\tan y = 0$ if $y = 0$, i.e. $x = \frac{\pi}{8}$.
   Then must have $1 - 2t - t^2 = 0$ at $x = \frac{\pi}{8}$, as required.
   The roots are $-1 \pm \sqrt{2}$. **[3]**
   As tan is increasing on $0 < x < \frac{\pi}{2}$ and $\tan 0 = 0$ we must have $\tan \frac{\pi}{8} = -1 + \sqrt{2}$. **[1]**
   [Total: 8]

3. (a) As $\sin 3\theta \cos 7\theta = \frac{1}{2}(\sin(10\theta) + \sin(-4\theta))$ we have

$$\int \sin 3\theta \cos 7\theta = -\frac{1}{2}\left[\frac{\cos 10\theta}{10} - \frac{\cos 4\theta}{4}\right] + C.$$

   [4]
   (b) Let $u = e^x$, and later $\sin\theta = u$. Then

$$\int e^x\sqrt{1 - e^{2x}}\,dx = \int \sqrt{1 - u^2}\,du = \int \cos^2\theta\,d\theta = \int \frac{1}{2}(1 + \cos 2\theta)\,d\theta$$
$$= \frac{1}{2}(\theta + \frac{1}{2}\sin 2\theta) + C = \frac{1}{2}(\sin^{-1}(e^x) + \frac{1}{2}\sin(2\sin^{-1}(e^x))) + C.$$

   [5]
   [Total: 9]

4. Claim that $\coth^2 x - 1 = \operatorname{cosech}^2 x$. Then check by expanding in terms of e^x. **[3]**
We have
$$3\coth x + \coth^2 x - 4 = 0$$
so
$$\coth x = \frac{-3 \pm 5}{4}.$$
[3]

But $|\coth x| > 1$ so $\coth x = -4$. Solving for x we obtain that
$$x = \frac{1}{2}\ln\left(\frac{3}{5}\right).$$
[3]
[Total: 9]

5. Let $f(x) = \frac{e^{2x}}{1-x}$, so $f(0) = 1$. Then
$$f'(x) = \frac{e^{2x}(3-2x)}{(1-x)^2} \quad \text{and } f'(0) = 3$$
$$f''(x) = \frac{e^{2x}(1-x)(4x^2-12x+10)}{(1-x)^4} \quad \text{and } f''(0) = 10.$$
[6]

Therefore
$$f(x) \approx f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 = 1 + 3x + 5x^2.$$
[2]
[Total: 8]

6. Check for $n = 1$. **[1]**
Suppose true for $n = k$, want also true for $n = k + 1$. **[1]**
Verify this. **[6]**
[Total: 8]