Maths for Actuarial Science Jan 09: Answers

1. We have
$$\frac{1+3x^2}{(1+x)^2(1+3x)} = \frac{A}{1+x} + \frac{B}{(1+x)^2} + \frac{C}{1+3x}.$$
Solving we find $A = 0$, $B = -2$, and $C = 3$. **[5]**
Hence the desired integral is
$$\int_1^3 \frac{3}{1+3x} - \frac{2}{(1+x)^2}\,dx = \left[\ln(1+3x) + \frac{2}{1+x}\right]_1^3 = \ln\left(\frac{5}{2}\right) - \frac{1}{2}.$$
[2]
[Total: 7]

2. (a) Expand and verify identity **[2]**
(b) $R\cos(2\theta - \alpha) = R\cos 2\theta \cos\alpha + R\sin 2\theta \sin\alpha$.
This implies that $R^2 = 32$ and hence $R = 4\sqrt{2}$
and $\tan\alpha = 1$ and hence $\alpha = \frac{\pi}{4}$. **[3]**
(c) Hence max value is $4\sqrt{2}$ and min is $-4\sqrt{2}$. **[1]**
(d) We have
$$4\sqrt{2}\cos(2\theta - \frac{\pi}{4}) + 5 = 9$$
and hence
$$\cos(2\theta - \frac{\pi}{4}) = \frac{1}{\sqrt{2}}.$$
[1]
This has general solution $2\theta - \frac{\pi}{4} = \pm\frac{\pi}{4} + 2n\pi$ with $n \in \mathbb{Z}$ and so $\theta = n\pi$ or $\theta = \frac{\pi}{4} + n\pi$ with $n \in \mathbb{Z}$. **[3]**
[Total: 10]

3. (a) These have centres $(2, 3)$ and $(8, 6)$ and radii 3 and 5. **[2]**
(b) The gradient is $\frac{6-3}{8-2} = \frac{1}{2}$ and so
$$y - 3 = \frac{1}{2}(x - 2).$$
This gives $2y = x + 4$. **[1]**
(c) Distance is $\sqrt{36+9} = \sqrt{45}$. Now $\sqrt{45} < 3 + 5 = 8$ and so the circles intersect. **[2]**
(d) The smallest circle has centre at the midpoint between the two centres, and radius half the distance between the centres. Midpoint is $(5, \frac{9}{2})$ and radius is $\frac{\sqrt{45}}{2}$ so equation is
$$(x-5)^2 + (y - \frac{9}{2})^2 = \frac{45}{4}.$$
[2]
[Total: 7]

4. (a) We have

$$\int x^2 \ln x \, dx = \frac{1}{3}x^3 \ln x - \int \frac{1}{3}x^3 \frac{1}{x} \, dx = \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C.$$

[3]

(b) Let $u = \sqrt{x}$ so $\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{2\sqrt{x}}$. Then

$$\int \frac{(1+\sqrt{x})^{\frac{1}{3}}}{\sqrt{x}} \, dx = \int 2(1+u)^{\frac{1}{3}} \, du = \frac{3}{2}(1+\sqrt{x})^{\frac{4}{3}} + C.$$

[4]

(c) We have

$$\int \frac{3}{x^2+2x+5} \, dx = \int \frac{3}{(x+1)^2+4} \, dx = \frac{3}{2} \tan^{-1}\left(\frac{x+1}{2}\right) + C.$$

[4]
[Total: 11]

5. (a) Let $y = \cosh(x)$ with $x \geq 0$ and so $x = \cosh^{-1} y$. Then $2y = e^x + e^{-x}$. Multiplying by e^x we have $e^{2x} - 2ye^x + 1 = 0$.
Solving for e^x we obtain

$$e^x = y \pm \sqrt{y^2 - 1}.$$

As $y \geq 1$ we have that $(y-1)^2 < y^2 - 1$ and hence $\sqrt{y^2-1} > y - 1$. For x to be non-negative we thus require $x = \ln(y + \sqrt{y^2-1})$. **[4]**
(b) Verify that $\cosh^2 x + \sinh^2 x = \cosh(2x)$. **[3]**
We have $\cosh 2x = \cosh 7$ if and only if $2x = \pm 7$, so the solutions are $x = \pm\frac{7}{2}$. **[2]**
[Total: 9]

6. Let $f(x) = \frac{e^{-x}}{1+x}$, so $f(0) = 1$. Then

$$f'(x) = \frac{-e^{-x}(2+x)}{(1+x)^2} \quad \text{and } f'(0) = -2$$

$$f''(x) = \frac{e^{-x}(x^3 + 5x^2 + 9x + 5)}{(1+x)^4} \quad \text{and } f''(0) = 5.$$

[4]

Therefore

$$f(x) \approx f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 = 1 - 2x + \frac{5}{2}x^2.$$

[2]
[Total: 6]