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CITY UNIVERSITY

London

BSc Degrees in Mathematical Science
Mathematical Science with Statistics
Mathematical Science with Computer Science
Mathematical Science with Finance and Economics
MMath Degrees in Mathematical Science

PART III EXAMINATION

Dynamical Systems

20-th of May 2004 9:00 am – 11:00 am

Time allowed: 2 hours

Full marks may be obtained for correct answers to THREE of the FIVE questions.

If more than THREE questions are answered, the best THREE marks will be credited.

Turn over . . .

1. Consider the following second order differential equation

$$\ddot{x} + \dot{x} + \alpha x^3 = 0 \qquad \text{for} \quad \alpha \in \mathbb{R}^+ \ .$$

(*i*) Find a suitable transformation of variables which changes the equation into a system of two first order differential equations.

(*ii*) Determine the fixed point of the system. State the linearization theorem and judge whether it is possible to draw conclusions from it concerning the stability of the fixed point.

(*iii*) Take the function

$$V(x_1, x_2) = \beta x_1^4 + \gamma x_2^2 \qquad \text{for} \quad \beta, \gamma \in \mathbb{R}^+$$

as a candidate for a Lyapunov function. Find a relation between the parameters α, β and γ such that $V(x_1, x_2)$ is a weak Lyapunov function. State the Lyapunov stability theorem and deduce from it the stability properties for the fixed point.

(*iv*) State the extension of the Lyapunov stability theorem and deduce from it the stability properties for the fixed point.

2. Consider the dynamical system of the form

$$\begin{aligned} \dot{x}_1 &= x_2 + x_1(\alpha - \beta x_1^2 - \beta x_2^2) \qquad \text{for} \quad \alpha, \beta \in \mathbb{R}^+, \ |\beta - \alpha| < 2 \\ \dot{x}_2 &= -x_1 + \beta x_2(1 - x_1^2 - x_2^2) \ . \end{aligned}$$

(*i*) Determine the nature of the fixed point at the origin.

(*ii*) Change the variables of the system to polar coordinates, using the conventions $x_1 = r\cos\vartheta$ and $x_2 = r\sin\vartheta$. Deduce from the equation for $\dot{\vartheta}$ that the origin is the only fixed point.

(*iii*) State the Poincaré-Bendixson theorem. Take from now on $\alpha = 2$ and $\beta = 3$ and employ the theorem to conclude that the system has at least one limit cycle in the annular region

$$\mathcal{D} = \left\{ (r, \vartheta) : 1/\sqrt{3} \le r \le 2 \right\} \ .$$

(*iv*) Determine some values $\mathrm{r}_{\min} = 1/\sqrt{3}$ and $\mathrm{r}_{\max} = 2$ such that the above conclusions also hold in the smaller annular region

$$\tilde{\mathcal{D}} = \{ (r, \vartheta) : r_{\min} \le r \le r_{\max} \} \ .$$

Turn over ...

3. Consider the dynamical system of the form

$$\begin{aligned}\dot{x}_1 &= (\lambda - 1)x_1 + x_2 + \lambda x_1^2 + 2x_1x_2 + x_1^2x_2 \\ \dot{x}_2 &= -\lambda x_1 - x_2 - 2x_1x_2 - \lambda x_1^2 - x_1^2x_2 \ .\end{aligned}$$

(i) Determine the nature of the fixed point at the origin for the linearized system depending on the values of λ. Can the linearization theorem be applied for $\lambda = 2$?

(ii) Use the stability index

$$I = \omega\left(Y^1_{111} + Y^1_{122} + Y^2_{112} + Y^2_{222}\right) + Y^1_{11}(Y^2_{11} - Y^1_{12}) + Y^2_{22}(Y^2_{12} - Y^1_{22}) + Y^2_{11}Y^2_{12} - Y^1_{22}Y^1_{12}$$

to argue that the origin is asymptotically stable for $\lambda = 2$, where the abbreviations $Y^i_{jk} = \partial^2 Y_i/\partial y_j \partial y_k$, $Y^i_{jkl} = \partial^3 Y_i/\partial y_j \partial y_k \partial y_l$ have been used. Carry out a similarity transformation on the Jacobian matrix, which brings it into the Jordan normal form

$$J = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \qquad \text{for } \omega \in \mathbb{R}^+ \ .$$

The $\vec{y}$ are the variables related to $\vec{x}$ through the similarity transformation.

(iii) State the Hopf bifurcation theorem and use it to prove that for $\lambda = 2$ the system possesses a Hopf bifurcation.

4. Consider the two dynamical systems of the form

$$\begin{aligned}\dot{x}_1 &= x_2 \qquad & \dot{x}_2 &= -x_1^4 + x_1 \ , \\ \dot{x}_1 &= x_1^2 + x_2 \qquad & \dot{x}_2 &= x_1^3 + 2x_2 \ .\end{aligned}$$

(i) Provide an argument (without explicit proof), which confirms that the first system is a set of equations of motion for a Hamiltonian system, whereas the second is not.

(ii) Find the Hamiltonian function and confirm that the system is also a potential system. Determine the potential and sketch it. Exploit the fact that the system is a potential system to deduce the position and the nature of the fixed points.

Turn over ...

(*iii*) Sketch the phase portrait by drawing some representative trajectories. Determine the equation for the separatrices, include it in the phase portrait and indicate the regions where $\dot{x}_1 > 0$ and $\dot{x}_2 > 0$.

5. Consider the following difference equation

$$x_{n+1} = F(x_n) = \lambda(1 - \alpha x_n^2) \qquad \text{for } \alpha, \lambda \in \mathbb{R}^+$$

α is taken to be a fixed constant and λ the bifurcation parameter.

(*i*) Determine the nature of the fixed points and their stability depending on the values of λ.

(*ii*) From the defining relation for a 2-cycle derive that they are determined by the solutions of

$$1 - \alpha x \lambda - \alpha \lambda^2 + \alpha^2 \lambda^2 x^2 = 0 \ .$$

Take from now on $\alpha = 1/2$ and argue that the existence of a 2-cycle requires $\lambda > \sqrt{3/2}$.

(*iii*) Show that the domain of stability is $\sqrt{3/2} < \lambda < \sqrt{5/2}$.

(*iv*) Sketch the bifurcation diagram.

Internal Examiner:	Dr. A. Fring
External Examiners:	Professor M.A. O'Neill
	Professor D.J. Needham