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CITY UNIVERSITY

London

BSc Degrees in Mathematical Science
Mathematical Science with Statistics
Mathematical Science with Computer Science
Mathematical Science with Finance and Economics
MMath Degrees in Mathematical Science

Part III Examination

Dynamical Systems

??-th of May 2005 9:00 am – 11:00 am

Time allowed: 2 hours

Full marks may be obtained for correct answers to THREE of the FIVE questions.

If more than THREE questions are answered, the best THREE marks will be credited.

Turn over . . .

1. Consider the dynamical system of the form

$$\dot{x}_1 = -x_1 - x_2^3 \qquad \text{and} \qquad \dot{x}_2 = x_1 .$$

(*i*) Show that the origin is the only fixed point of the system.

(*ii*) State the linearization theorem and judge whether it is possible to draw conclusions from it concerning the stability of the fixed point.

(*iii*) State a theorem which serves as a sufficient condition to decide when a dynamical system does not possess a limit cycles. Apply the theorem to the above system and show that is has no limit cycles.

(*iv*) State the Lyapunov stability theorem and deduce from it the stability properties for the fixed point by showing that the function

$$V(x_1, x_2) = 2x_1^2 + 2x_1x_2 + x_2^2 + x_2^4$$

is a strong Lyapunov function for the above system.

2. Consider the dynamical system of the form

$$\begin{aligned} \dot{x}_1 &= x_2 + x_1(4 - 5x_1^2 - 5x_2^2) \\ \dot{x}_2 &= -x_1 + 5x_2(1 - x_1^2 - x_2^2) \ . \end{aligned}$$

(*i*) Determine the nature of the fixed point at the origin.

(*ii*) Change the variables of the system to polar coordinates, using the conventions $x_1 = r\cos\vartheta$ and $x_2 = r\sin\vartheta$. Deduce from the equation for $\dot{\vartheta}$ that the origin is the only fixed point.

(*iii*) State the Poincaré-Bendixson theorem. Employ the theorem to conclude that the system has at least one limit cycle in the annular region

$$\mathcal{D} = \{(r, \vartheta) : 1/2 \leq r \leq 2\} \ .$$

(*iv*) Determine some values $\mathrm{r_{min}}$ and $\mathrm{r_{max}}$ such that the above conclusions also hold in the smaller annular region

$$\tilde{\mathcal{D}} = \{(r, \vartheta) : r_{\min} \leq r \leq r_{\max}\} \ .$$

Turn over ...

3. Consider the dynamical system of the form

$$\begin{aligned}\dot{x}_1 &= (\lambda-1)x_1 + 4x_2 + \frac{\lambda}{2}x_1^2 + 4x_1x_2 + x_1^2x_2 \\ \dot{x}_2 &= -\lambda x_1 - 4x_2 - 4x_1x_2 - \frac{\lambda}{2}x_1^2 - x_1^2x_2 \ .\end{aligned}$$

(*i*) Determine the nature of the fixed point at the origin for the linearized system depending on the values of the bifurcation parameter λ. Is it possible to apply the linearization theorem for $\lambda = 5$?

(*ii*) Use the stability index

$$\begin{aligned}I &= \omega\left(Y^1_{111} + Y^1_{122} + Y^2_{112} + Y^2_{222}\right) \\ &\quad + Y^1_{11}(Y^2_{11} - Y^1_{12}) + Y^2_{22}(Y^2_{12} - Y^1_{22}) + Y^2_{11}Y^2_{12} - Y^1_{22}Y^1_{12}\end{aligned}$$

to argue that the origin is asymptotically stable for $\lambda = 5$, where the abbreviations $Y^i_{jk} = \partial^2 Y_i/\partial y_j \partial y_k$, $Y^i_{jkl} = \partial^3 Y_i/\partial y_j \partial y_k \partial y_l$ have been used. A similarity transformation on the Jacobian matrix A brings it into the Jordan normal form

$$J = U^{-1}AU = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \qquad \text{for } \ \omega \in \mathbb{R}^+, U = \begin{pmatrix} 2 & -4 \\ 0 & 5 \end{pmatrix} .$$

The $\vec{y}$ variables are related to the $\vec{x}$ variables as $\vec{x} = U\ \vec{y}$, such that the above system $\dot{x}_i = X_i(\vec{x})$ changes to $\dot{y}_i = Y_i(\vec{y})$ with

$$\begin{aligned}\dot{y}_1 &= Y_1(\vec{y}) = -3y_1^2 - 6y_1^2y_2 + 12y_2^2 + 24y_1y_2^2 - 24y_2^3, \\ \dot{y}_2 &= Y_2(\vec{y}) = -2y_1^2 - 4y_1^2y_2 + 8y_2^2 + 16y_1y_2^2 - 16y_2^3.\end{aligned}$$

(*iii*) State the Hopf bifurcation theorem and use it to prove that for $\lambda = 5$ the system possesses a Hopf bifurcation.

Turn over ...

4. Consider the two dynamical systems of the form

$$
\begin{array}{llll}
a) & \dot{x}_1 = x_2 & & \dot{x}_2 = -2\cos(2x_1) \\
b) & \dot{x}_1 = 2x_1^3 + \cos(x_2) & & \dot{x}_2 = 3x_1 + \cos(x_2)
\end{array}
$$

(*i*) Show that the system *a*) is a set of equations of motion for a Hamiltonian system, whereas the system *b*) is not.

(*ii*) For the system *a*) derive the Hamiltonian function and confirm that the system is also a potential system. Set the ground state energy to zero.

(*iii*) Find all fixed points of the system a) in the range $0 \leq x_1 \leq 2\pi$ and determine their nature.

(*iv*) Determine the equation for the separatrices of the system a) and sketch the phase portrait in the range $0 \leq x_1 \leq 2\pi$ by drawing some representative trajectories. Include the seperatrix in your phase portrait and indicate the area of bounded motion. Provide a reasoning for the choice of direction on the trajectories.

5. Consider the following difference equation

$$x_{n+1} = F(x_n) = \lambda^2 x_n + (1-\lambda)x_n^2 \qquad \text{for} \quad \lambda > 1.$$

with λ taken to be the bifurcation parameter.

(*i*) Find the fixed points and determine the stability properties depending on the values of λ.

(*ii*) State the condition which determines the existence of a 2-cycle. Show that 2-cycles are determined by the solutions of

$$x^2(\lambda-1)^2 + x(1-\lambda)(1+\lambda^2) + (1+\lambda^2) = 0 \; .$$

Find the solution of this equation and use it to argue that the existence of a 2-cycle requires $\lambda > \sqrt{3}$.

(*iii*) State the stability condition for a 2-cycle and employ it to decide whether the two cycles are stable or not.

Internal Examiner:	Dr. A. Fring
External Examiners:	Professor M.A. O'Neill
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