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CITY UNIVERSITY

London

BSc Degrees in Mathematical Science
Mathematical Science with Statistics
Mathematical Science with Computer Science
Mathematical Science with Finance and Economics
MMath Degrees in Mathematical Science

PART III EXAMINATION

Dynamical Systems

12-th of January 2006 14:30 am – 16:30 am

Time allowed: 2 hours

Full marks may be obtained for correct answers to THREE of the FIVE questions.

If more than THREE questions are answered, the best THREE marks will be credited.

Turn over . . .

1. Consider the following second order differential equation

$$\ddot{x} + \dot{x} + x - \dot{x}x^2 = 0.$$

(i) Find a suitable transformation of variables which changes this equation into a system of two first order differential equations.

(ii) Determine all fixed points of the system. State the linearization theorem and judge whether it is possible to draw conclusions from it concerning the stability of the fixed points and the qualitative behaviour of the system near the fixed points.

(iii) State a theorem which serves as a sufficient condition to decide when a dynamical system does not possess any limit cycles and one theorem which is a necessary condition for a limit cycle. Based on these theorems produce a sketch of a phase portrait which indicates possible limit cycles.

(iv) State the Lyapunov stability theorem and deduce from it the stability properties for the fixed point first by showing that the function

$$V(x_1, x_2) = x_1^2 + x_2^2$$

is a weak Lyapunov function for the above system. Use the extension of the Lyapunov theorem to draw a stronger conclusion. Determine the domain of stability.

2. Consider the dynamical system of the form

$$\begin{aligned}\dot{x}_1 &= -x_2 + 6x_1(1 - x_1^2 - x_2^2) \\ \dot{x}_2 &= x_1 + x_2(5 - 6x_1^2 - 6x_2^2)\ .\end{aligned}$$

(*i*) Determine the nature of the fixed point at the origin.

(*ii*) Change the variables of the system to polar coordinates, using the conventions $x_1 = r\cos\vartheta$ and $x_2 = r\sin\vartheta$. Deduce from the equation for $\dot{\vartheta}$ that the origin is the only fixed point.

(*iii*) State the Poincaré-Bendixson theorem. Employ the theorem to conclude that the system has at least one limit cycle in the annular region

$$\mathcal{D} = \{(r,\vartheta) : 1/2 \leq r \leq 3\}\ .$$

(*iv*) Determine some values $\mathrm{r}_{\min} > 1/2$ and $\mathrm{r}_{\max} < 3$ such that the above conclusions also hold in the smaller annular region

$$\tilde{\mathcal{D}} = \{(r,\vartheta) : r_{\min} \leq r \leq r_{\max}\}\ .$$

3. (*i*) Describe what is meant by a bifurcation. For a one dimensional dynamical system give a precise definition of a) a turning point b) a transcritical bifurcation and c) a pitchfork bifurcation.

(*ii*) Provide for each of the definitions in (*i*) a concrete example together with a sketch of its bifurcation diagram.

(*iii*) What is meant by a Hopf bifurcation?

(*iv*) For the following system

$$\dot{r} = \lambda r(r-\lambda)^2 \qquad \text{and} \qquad \dot{\vartheta} = 1$$

sketch the phase portrait for positive λ and determine the α and ω limit sets. Sketch the bifurcation diagram in the (r,λ)-plane with $\lambda \in \mathbb{R}$ being the bifurcation parameter. Decide which type of bifurcation occurs at the point $(r,\lambda) = (0,0)$.

4. Consider the two dynamical systems

$$
\begin{array}{llll}
a) & \dot{x}_1 = x_2 & & \dot{x}_2 = -x_1^3 + x_1 \\
b) & \dot{x}_1 = x_1^3 + \kappa x_1 \cos(x_2) & & \dot{x}_2 = -3x_1^2 x_2 + 2\sin(x_2)
\end{array}
$$

(*i*) Show that the system *a*) is a set of equations of motion for a Hamiltonian system. Determine the value for the parameter κ such the system *b*) also becomes a Hamiltonian system.

(*ii*) For the system *a*) derive the Hamiltonian function and confirm that the system is also a potential system. Set the minimum of the potential to zero.

(*iii*) Find all fixed points of the system a) and determine their nature.

(*iv*) Sketch the potential, determine the equation for the separatrices of the system a) and sketch the phase portrait by drawing some representative trajectories. Include the separatrix in your phase portrait and indicate the area of bounded motion. Provide a reasoning for the choice of direction on the trajectories.

5. Consider the following difference equation

$$x_{n+1} = F(x_n) = \lambda x_n(2 - x_n) \qquad \text{for} \quad \lambda > 1/2.$$

with λ taken to be the bifurcation parameter.

(*i*) Find the fixed points and show that one of them is always unstable and the other is stable for $\lambda < 3/2$.

(*ii*) State the condition which determines the existence of a 2-cycle. Show that 2-cycles are determined by the solutions of

$$1 + 2\lambda - x\lambda - 2x\lambda^2 + x^2\lambda^2 = 0 \; .$$

Find the solution of this equation and use it to argue that the existence of a 2-cycle requires $\lambda > 3/2$.

(*iii*) State the stability condition for a 2-cycle and employ it to decide in which range for λ the 2-cycles are stable.

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