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CITY UNIVERSITY

London

BSc Degrees in Mathematical Science
Mathematical Science with Statistics
Mathematical Science with Computer Science
Mathematical Science with Finance and Economics
MMath Degrees in Mathematical Science

PART III EXAMINATION

Dynamical Systems

9th of January 2007 14:30 am – 16:30 am

Time allowed: 2 hours

Full marks may be obtained for correct answers to THREE of the FIVE questions.

If more than THREE questions are answered, the best THREE marks will be credited.

Turn over ...

1. Consider the following dynamical system

$$\begin{aligned}\dot{x}_1 &= -8x_1 - x_1x_2^2 - 3x_2^3 \\ \dot{x}_2 &= 2x_1x_2^2 + 2x_1^2x_2 \ .\end{aligned}$$

(i) Determine all fixed points of the system. State the linearization theorem and judge whether it is possible to draw conclusions from it concerning the stability of the fixed points and the qualitative behaviour of the system near the fixed points.

(ii) State the Lyapunov stability theorem and deduce from it the stability properties for the fixed point at the origin by showing that the function

$$V(x_1, x_2) = 2x_1^2 + 3x_2^2$$

is a weak Lyapunov function for the above system.

(iii) State the extension of the Lyapunov stability theorem and argue that the origin is an asymptotically stable fixed point.

(iv) Show that the domain of Lyapunov stability is bounded by the ellipse

$$2x_1^2 + 3x_2^2 = 12.$$

Turn over . . .

2. (*i*) State a theorem which serves as a sufficient condition to decide when a dynamical system does not possess any limit cycles.

(*ii*) State a necessary condition for the existence of a limit cycle.

(*iii*) Based on the theorems in (*i*) and (*ii*) prove that none of the following dynamical systems has any limit cycle. Support the cases *c*) - *e*) with some sketches of the corresponding phase portrait.

a)

$$\dot{x}_1 = 1 + x_2^2 - \exp(x_1 x_2) \qquad \text{and} \qquad \dot{x}_2 = x_1 x_2 + \cos^2(x_2)$$

b)

$$\dot{x}_1 = 3x_2 + x_1^3 \qquad \text{and} \qquad \dot{x}_2 = x_1 + x_2 + x_2^3$$

c)

$$\dot{x}_1 = x_1 - x_2^2(1 + x_1^3) \qquad \text{and} \qquad \dot{x}_2 = x_1^5 - x_2$$

d)

$$\dot{x}_1 = 4x_1 - 2x_1^2 - x_2^2 \qquad \text{and} \qquad \dot{x}_2 = x_1 + x_2 x_1^2$$

e)

$$\dot{x}_1 = x_1^2 - x_2 - 1 \qquad \text{and} \qquad \dot{x}_2 = x_1 x_2 - 2x_2$$

Turn over . . .

3. Consider the dynamical system of the form

$$\begin{aligned}\dot{x}_1 &= x_2 + x_1(4 - 5x_1^2 - 5x_2^2)\\ \dot{x}_2 &= -x_1 + 5x_2(1 - x_1^2 - x_2^2)\ .\end{aligned}$$

(*i*) By linearizing the system determine the nature of the fixed point at the origin.

(*ii*) Change the variables of the system to polar coordinates, using the conventions $x_1 = r\cos\vartheta$ and $x_2 = r\sin\vartheta$. Deduce from the equation for $\dot{\vartheta}$ that the origin is the only fixed point of the system.

(*iii*) State the Poincaré-Bendixson theorem. Employ the theorem to conclude that the system has at least one limit cycle in the annular region

$$\mathcal{D} = \{(r,\vartheta) : 1/2 \le r \le 2\}\ .$$

(*iv*) Determine some values $\mathrm{r}_{\min} > 1/2$ and $\mathrm{r}_{\max} < 2$ such that the above conclusions also hold in the smaller annular region

$$\tilde{\mathcal{D}} = \{(r,\vartheta) : r_{\min} \le r \le r_{\max}\}\ .$$

Turn over . . .

4. Consider the two dynamical systems

$$
\begin{aligned}
a)\qquad \dot{x}_1 &= x_2 & \dot{x}_2 &= x_1 + x_1^2 \\
b)\qquad \dot{x}_1 &= x_2 - x_2^2 + x_1^2 & \dot{x}_2 &= -x_1 - \lambda x_1 x_2
\end{aligned}
$$

(*i*) Provide a criterium which can be used to establish whether a system is Hamiltonian or not. Use this criterium to show that the system *a*) is a set of equations of motion for a Hamiltonian system. Determine the value for the parameter λ such that the system *b*) is also a Hamiltonian system.

(*ii*) Prove that a Hamiltonian system is preserved along trajectories. What does this imply for the form of the trajectories?

(*iii*) For the system *a*) derive the Hamiltonian function and confirm that the system is also a potential system. Fix the potential such that it vanishes at the origin.

(*iv*) Find all fixed points of the system *a*) and determine their nature by exploiting the fact that the system is a potential system.

(*v*) Sketch the potential, determine the equation for the separatrices of the system *a*) and sketch the phase portrait by drawing some representative trajectories. Include the separatrix in your phase portrait and indicate the area of bounded motion. Provide a reasoning for the choice of direction on the trajectories.

Turn over ...

5. Consider the following difference equation

$$x_{n+1} = F(x_n) = \lambda x_n(1 - x_n) \qquad\qquad \text{for } \lambda \in \mathbb{R}^+.$$

λ is taken to be the bifurcation parameter.

(i) Depending on the values of λ, determine the nature of the fixed points and their stability.

(ii) State the condition which determines the existence of a 2-cycle. Show that 2-cycles for the above system are determined by the solutions of the equation

$$2x^2\lambda^2 - x(\lambda + \lambda^2) + 1 + \lambda = 0 \ .$$

Determine the solution of this equation and use it to argue that the existence of a 2-cycle requires $\lambda \geq 3$.

(iii) Determine the domain of stability for the 2-cycle and sketch the corresponding bifurcation diagram.

(iv) Use the following variable transformation

$$x \to \sin^2 \vartheta$$

to show that the system becomes the tent map for when $\lambda = 4$.

Internal Examiner:	Dr. A. Fring
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