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CITY UNIVERSITY

London

BSc Degrees in Mathematical Science
Mathematical Science with Statistics
Mathematical Science with Computer Science
Mathematical Science with Finance and Economics
MMath Degrees in Mathematical Science

PART III EXAMINATION

Dynamical Systems

16th of January 2008 14:30 am – 16:30 am

Time allowed: 2 hours

Full marks may be obtained for correct answers to THREE of the FIVE questions.

If more than THREE questions are answered, the best THREE marks will be credited.

Turn over . . .

1. A competing species model is given by the following dynamical system

$$\begin{aligned}\dot{x}_1 &= x_1(\gamma_1 - s_1 x_1 - \varepsilon_1 x_2)\\ \dot{x}_2 &= x_2(\gamma_2 - s_2 x_2 - \varepsilon_2 x_1)\end{aligned} \qquad \text{with } \gamma_i, s_i, \varepsilon_i \in \mathbb{R}^+, i = 1, 2,$$

where x_i is the population of the species i, γ_i denotes the growth rate of the species i, and γ_i/s_i is the saturation level for the population of the species i.

(*i*) Explain briefly the meaning of the parameters ε_i. Which scenario is described when $\varepsilon_1 = \varepsilon_2 = 0$?

(*ii*) Take from now on the parameters of the model to be fixed as

$$\gamma_1 = s_1 = s_2 = \varepsilon_1 = 1, \qquad \varepsilon_2 = 2, \qquad \text{and} \qquad \gamma_2 = \frac{3}{2}.$$

Using these values determine all fixed points for the competing species model.

(*iii*) Decide for each of the fixed points whether it is possible for the two species to coexist.

(*iv*) State the linearization theorem and decide whether it can be applied at each of the fixed points. Determine the nature of the fixed points.

(*v*) Compute the isoclines for the model and use this information to sketch the phase portrait. Indicate the regions in which $\dot{x}_1 > 0$ and $\dot{x}_2 > 0$. Precise local phase portraits at each fixed point are not required.

Turn over ...

2. Consider the dynamical system of the form

$$\begin{aligned}\dot{x}_1 &= -3x_1 - \frac{1}{2}x_2^3 + x_1 x_2^2 \\ \dot{x}_2 &= 2x_1 x_2^2 + 2x_2 x_1^2\end{aligned}$$

(*i*) State the Lyapunov stability theorem and the definitions for a weak and strong Lyapunov function. Show that the function

$$V(x_1, x_2) = 8x_1^2 + 2x_2^2$$

is a weak Lyapunov function for the system specified above. Deduce from this the stability properties of the fixed point.

(*ii*) Determine the length of the major and minor axis of the ellipse which confines the maximal domain of stability.

(*iii*) State the extension of the Lyapunov stability theorem and conclude from it that the fixed point is asymptotically stable.

3. Consider the dynamical system of the form

$$\begin{aligned}\dot{x}_1 &= 25x_1 - x_2 - x_1(x_1^2 + x_2^2) \\ \dot{x}_2 &= x_1 + 25x_2 - x_2(x_1^2 + x_2^2)\ .\end{aligned}$$

(*i*) Change the variables of the system to polar coordinates, using the conventions $x_1 = r\cos\vartheta$ and $x_2 = r\sin\vartheta$. Deduce from the equation for $\dot{\vartheta}$ that the origin is the only fixed point of the system.

(*ii*) State the Poincaré-Bendixson theorem. Employ the theorem to conclude that the system has at least one limit cycle in the annular region

$$\mathcal{D} = \{(r, \vartheta) : 4 \leq r \leq 6\}\ .$$

(*iii*) Sketch the phase portrait for the above system. Determine exactly the limit cycle in $\mathcal{D}$.

(*iv*) State the definitions for the α-limit set and the ω-limit set and determine them thereafter. Decide whether the limit cycle in (*iii*) is stable, unstable or semistable.

(*v*) State a sufficient condition for the non-existence of a limit cycle in a particular domain.

Turn over ...

4. (*i*) Provide a criterium which can be used to establish whether a dynamical system with one degree of freedom is Hamiltonian or not.

(*ii*) Compute the equations of motion corresponding to the Hamiltonian

$$H(x_1, x_2) = \frac{1}{2}x_2^2 + \frac{x_1}{1 + x_1^4}$$

(*iii*) Exploit the fact that the Hamiltonian is a potential system to find and classify all fixed points for the dynamical system computed in (*ii*).

(*iv*) Sketch a graph of the potential and use it to construct a phase portrait for the dynamical system computed in (*ii*). Compute the separatrices and include them into your phase portrait.

(*v*) Identify the region in phase space in which the motion is periodic and find the period of such a motion. You may leave your answer in form of a definite integral.

5. Consider the following difference equation

$$x_{n+1} = F(x_n) = \lambda x_n(4 - x_n) \qquad \text{for } \lambda \in \mathbb{R}^+.$$

λ is taken to be the bifurcation parameter.

(*i*) Depending on the values of λ, determine the nature of the fixed points and their stability.

(*ii*) State the condition which determines the existence of a 2-cycle. Show that 2-cycles for the above system are determined by the solutions of the equation

$$x^2\lambda^2 - 4x\lambda^2 - x\lambda + 4\lambda + 1 = 0 \ .$$

Compute the solution of this equation and use it to argue that the existence of a 2-cycle requires $\lambda \geq 3/4$.

(*iii*) Determine the domain of stability for the 2-cycle and sketch the corresponding bifurcation diagram.

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