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CITY UNIVERSITY

London

BSc Degrees in Mathematical Science
Mathematical Science with Statistics
Mathematical Science with Computer Science
Mathematical Science with Finance and Economics
MMath Degrees in Mathematical Science

PART III EXAMINATION

Dynamical Systems

14th of January 2009 14:30 am – 16:30 am

Time allowed: 2 hours

Full marks may be obtained for correct answers to THREE of the FIVE questions.

If more than THREE questions are answered, the best THREE marks will be credited.

Turn over . . .

1. Consider the dynamical system of the form

$$\dot{x}_1 = x_1 x_2^2 - 9x_1 - 16x_2^3 \qquad \dot{x}_2 = 4x_1 x_2^2 + 2x_2 x_1^2.$$

(*i*) State the Lyapunov stability theorem and the definitions for a weak and strong Lyapunov function. Show that the function

$$V(x_1, x_2) = 4x_1^2 + 16x_2^2$$

is a weak Lyapunov function for the dynamical system specified above. Deduce from this the stability properties of the fixed point.

(*ii*) Determine the length of the major and minor axis of the ellipse which confines the maximal domain of stability.

(*iii*) State the extension of the Lyapunov stability theorem and conclude from it that the fixed point is asymptotically stable.

Turn over . . .

2. Consider the dynamical system of the form

$$\dot{x}_1 = x_1 - x_2 - x_1^3, \qquad \dot{x}_2 = x_1 + x_2 - x_2^3$$

(i) Using the conventions $x_1 = r\cos\vartheta$ and $x_2 = r\sin\vartheta$ change the variables of the system to polar coordinates. Deduce from the equation for $\dot{\vartheta}$ and $\dot{r}$ that the origin is the only fixed point of the system.

(ii) State the Poincaré-Bendixson theorem and employ it to conclude that the above system has at least one limit cycle in the annular region

$$\mathcal{D} = \left\{ (r,\vartheta) : \frac{1}{3} \leq r \leq 2 \right\} .$$

(iii) State Bendixson's criterium and a theorem which provides a relation betwen fixed points and limit cycles. Use these criteria to show that the system

$$\dot{x}_1 = x_1^2 - x_2 - 1, \qquad \dot{x}_2 = x_1 x_2 - 2x_2$$

can not possess a limit cycle. Sketch a phase portrait with the relevant information, which justifies your conclusions.

Hint: You may use the identities: $4(\cos^4\vartheta + \sin^4\vartheta) = 3 + \cos(4\vartheta)$, $\sin(2\vartheta) = 2\cos\vartheta\sin\vartheta$, $\cos(2\vartheta) = \cos^2\vartheta - \sin^2\vartheta$.

Turn over ...

3. (*i*) Describe what is meant by a bifurcation. For a one dimensional dynamical system provide a precise definition of a) a turning point b) a transcritical bifurcation and c) a pitchfork bifurcation.

(*ii*) Consider the dynamical system

$$\begin{aligned} \dot{x}_1 &= 7x_2 \\ \dot{x}_2 &= -(x_1^2 - \lambda)x_2 - 7x_1 - 2x_1^3 \end{aligned}$$

with $\lambda \in \mathbb{R}$ being a bifurcation parameter. Use the stability index

$$\begin{aligned} I &= \omega\left(Y^1_{111} + Y^1_{122} + Y^2_{112} + Y^2_{222}\right) + Y^1_{11}Y^2_{11} - Y^1_{11}Y^1_{12} + Y^2_{11}Y^2_{12} \\ &\quad + Y^2_{22}Y^2_{12} - Y^1_{22}Y^1_{12} - Y^1_{22}Y^2_{22} \end{aligned}$$

to argue that the origin is asymptotically stable for $\lambda = 0$. We abbreviated $Y^i_{jk} = \partial^2 Y_i/\partial y_j \partial y_k$ and $Y^i_{jkl} = \partial^3 Y_i/\partial y_j \partial y_k \partial y_l$.

(*iii*) State the Hopf bifurcation theorem and use it to prove that the system possesses a Hopf bifurcation for $\lambda = 0$.

(*iv*) For the following system

$$\dot{r} = \lambda r(r - 2\lambda)^2 \qquad \text{and} \qquad \dot{\vartheta} = -1$$

sketch the phase portrait for positive λ and determine the α and ω limit sets. Sketch the bifurcation diagram in the (r, λ)-plane with $\lambda \in \mathbb{R}$ being the bifurcation parameter. Decide which type of bifurcation occurs at the point $(r, \lambda) = (0, 0)$.

Turn over . . .

4. (*i*) Provide a definition for a Hamiltonian system in two dimensions.

(*ii*) Prove that any nondegenerate fixed point of a Hamiltonian system is either a saddle point or a centre. Provide a simple criterium involving derivatives of the Hamiltonian, which allows to decide which type of fixed point is realized.

(*iii*) Provide a definition for a potential system. Use the fact that the following system

$$H(x_1, x_2) = \frac{1}{2}x_2^2 + \kappa e^{-x_1} \sin x_1 \qquad \kappa \in \mathbb{R}$$

is a potential system to find and classify all its fixed points.

(*iv*) For a potential system derive the expression

$$T = 2\int_\alpha^\beta \frac{dx}{\sqrt{2[E - V(x)]}}$$

for the period of a motion around a centre. Explain in your derivation the meaning of the constants α, β and E.

(*v*) Given the initial condition $x_1 = 2^{3/4}$, $x_2 = 2$ compute the period T for the potential system

$$H(x_1, x_2) = \frac{1}{2}x_2^2 + \frac{1}{4}x_1^4.$$

Hint: You may use the integral $\int_0^1 dx/\sqrt{1 - x^4} = \sqrt{\pi}\Gamma(5/4)/\Gamma(3/4)$.

Turn over . . .

5. Consider the following difference equation

$$x_{n+1} = F(x_n) = x_n^2 + 2\lambda x_n - 15\lambda^2 \qquad \text{for } \lambda \in \mathbb{R}^+.$$

λ is taken to be the bifurcation parameter.

(i) Depending on the values of λ, determine the nature of the fixed points and their stability.

(ii) State the condition which determines the existence of a 2-cycle. Show that 2-cycles for the above system are determined by the solutions of the equation

$$1 + x + x^2 + 2\lambda + 2x\lambda - 15\lambda^2 = 0 \ .$$

Compute the solution of this equation and use it to argue that the existence of a 2-cycle requires $\lambda \geq 1/4$.

(iii) Determine the domain of stability for the 2-cycle.

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